ELECTROMAGNETIC FORCE LAW IN QUANTUM MECHANICS

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In classical electromagnetism, the force on a particle of charge q moving with velocity $\mathbf{v}$ through an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is

$$\mathbf{F}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) \tag{1}$$

Because this isn't expressible as the gradient of a potential V, we can't use the Schrödinger equation in the form where the kinetic energy is expressed in terms of the laplacian operator ∇^{2}.

The classical hamiltonian (something we won't derive here, although it comes out of classical mechanics and Maxwell's equations) can be written as

$$H=\frac{1}{2m}\left(\mathbf{p}-q\mathbf{A}\right)^{2}+q\varphi \tag{2}$$

where $\mathbf{A}$ is the magnetic vector potential (which means that $\mathbf{B}=\nabla\times\mathbf{A}$) and φ is the electric scalar potential. The electric field is written as $\mathbf{E}=-\nabla\varphi-\partial\mathbf{A}/\partial t$. (If all of this looks unfamiliar, you can just accept it for now. Hopefully I'll eventually get to the stage in electromagnetism where I can fill in the background for this.)

Making the usual quantum substitution of $\mathbf{p}=-i\hbar\nabla$, the Schrödinger equation then becomes

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla-q\mathbf{A}\right)^{2}+q\varphi\right]\Psi \tag{3}$$

We can begin the analysis of this equation by working out $d\langle\mathbf{r}\rangle/dt$. We have

$$\langle\mathbf{r}\rangle=\int\Psi^{*}\mathbf{r}\Psi d^{3}\mathbf{r} \tag{4}$$

Taking the time derivative, we get

$$\frac{d\langle\mathbf{r}\rangle}{dt} = \int \frac{\partial\mathbf{\Psi}^*}{\partial t}\mathbf{r}\mathbf{\Psi}d^3\mathbf{r} + \int \mathbf{\Psi}^*\mathbf{r}\frac{\partial\mathbf{\Psi}}{\partial t}d^3\mathbf{r} \tag{5}$$

$$= \frac{i}{\hbar}\int\left[\frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right]\mathbf{\Psi}^*\mathbf{r}\mathbf{\Psi}d^3\mathbf{r} - \tag{6}$$

$$\frac{i}{\hbar}\int\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right]\mathbf{\Psi}\mathbf{r}\mathbf{\Psi}^*d^3\mathbf{r} \tag{7}$$

We need to work out the effect of the Hamiltonian operator on the wave function, so consider the first integrand above

$$\left[\frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right]\mathbf{\Psi}^* = \frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)\mathbf{\Psi}^* + q\varphi\mathbf{\Psi}^* \tag{8}$$

$$= \frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{-i}\nabla\mathbf{\Psi}^* - q\mathbf{A}\mathbf{\Psi}^*\right) + q\varphi\mathbf{\Psi}^* \tag{9}$$

$$= \frac{1}{2m}\left(-\hbar^2\nabla^2\mathbf{\Psi}^* + \frac{\hbar q}{i}\nabla\cdot(\mathbf{\Psi}^*\mathbf{A}) + \frac{\hbar q}{i}\mathbf{A}\cdot\nabla\mathbf{\Psi}^* + q^2|\mathbf{A}|^2\mathbf{\Psi}^*\right) + q\varphi\mathbf{\Psi}^* \tag{10}$$

$$= \frac{1}{2m}\left(-\hbar^2\nabla^2\mathbf{\Psi}^* + \frac{\hbar q}{i}(2\mathbf{A}\cdot\nabla\mathbf{\Psi}^* + \mathbf{\Psi}^*\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\mathbf{\Psi}^*\right) + q\varphi\mathbf{\Psi}^* \tag{11}$$

where the last line uses

$$\nabla\cdot(\mathbf{\Psi}^*\mathbf{A}) = \mathbf{A}\cdot\nabla\mathbf{\Psi}^* + \mathbf{\Psi}^*\nabla\cdot\mathbf{A} \tag{12}$$

The second integrand is just the complex conjugate of the one we just worked out, so

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right]\mathbf{\Psi} = \frac{1}{2m}\left(-\hbar^2\nabla^2\mathbf{\Psi} + \frac{\hbar q}{-i}(2\mathbf{A}\cdot\nabla\mathbf{\Psi} + \mathbf{\Psi}\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\mathbf{\Psi}\right) + q\varphi\mathbf{\Psi} \tag{13}$$

Multiplying the first equation by $\mathbf{\Psi}$ and the second by $\mathbf{\Psi}^*$ and subtracting gives the integral required above. Note that all terms not involving a derivative cancel out in the subtraction.

$$\frac{d\langle\mathbf{r}\rangle}{dt} = \frac{i\hbar}{2m}\int(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*)\mathbf{r}d^3\mathbf{r} + \tag{14}$$
$$\frac{q}{m}\int(\Psi\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\mathbf{A}\cdot\nabla\Psi + |\Psi|^2\nabla\cdot\mathbf{A})\mathbf{r}d^3\mathbf{r} \tag{15}$$

Each of these integrals gives a vector due to the presence of **r** in the integrand. The easiest way to see the result is to consider the three components of the vector separately. Consider the first integrand. The x component is

$$\frac{d\langle x\rangle}{dt} = \frac{i\hbar}{2m}\int\left[x\Psi^*\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) - x\Psi\left(\frac{\partial^2\Psi^*}{\partial x^2} + \frac{\partial^2\Psi^*}{\partial y^2} + \frac{\partial^2\Psi^*}{\partial z^2}\right)\right]dxdydz \tag{16}$$

The terms involving derivatives with respect to x are identical to those found in the original derivation of the momentum operator in one dimension. The terms involving derivatives with respect to y or z all integrate to zero, since the integration by parts step involves either $\partial y/\partial x$ or $\partial z/\partial x$, both of which give zero. For example

$$\int y\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right)dx = \int y\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)dx \tag{17}$$
$$= y\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)\Big|_{x=-\infty}^{x=\infty} - \int\frac{\partial y}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)dx \tag{18}$$
$$= 0 \tag{19}$$

where the first term in the second line is zero due to the usual requirement that the wave function and its derivatives go to zero at infinity, and the second term is zero because $\partial y/\partial x = 0$.

Similar results occur in the other two components of **r**. Thus the first integral works out to

$$\frac{i\hbar}{2m}\int(\Psi^*\nabla^2\Psi - \Psi\nabla^2\Psi^*)\mathbf{r}d^3\mathbf{r} = \langle\mathbf{p}\rangle/m \tag{20}$$

To see the second integrand more clearly, we can write out the ∇ operations in terms of their components.

$$\Psi \mathbf{A} \cdot \nabla \Psi^* + \Psi^* \mathbf{A} \cdot \nabla \Psi + |\Psi|^2 \nabla \cdot \mathbf{A} = \Psi \left(A_x \frac{\partial \Psi^*}{\partial x} + A_y \frac{\partial \Psi^*}{\partial y} + A_z \frac{\partial \Psi^*}{\partial z} \right) + \tag{21}$$

$$\Psi^* \left(A_x \frac{\partial \Psi}{\partial x} + A_y \frac{\partial \Psi}{\partial y} + A_z \frac{\partial \Psi}{\partial z} \right) + \tag{22}$$

$$\Psi^* \Psi \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \tag{23}$$

$$= \frac{\partial (\Psi^* \Psi A_x)}{\partial x} + \frac{\partial (\Psi^* \Psi A_y)}{\partial y} + \frac{\partial (\Psi^* \Psi A_z)}{\partial z} \tag{24}$$

Using the same integration by parts technique as before, we get

$$\frac{q}{m} \int (\Psi \mathbf{A} \cdot \nabla \Psi^* + \Psi^* \mathbf{A} \cdot \nabla \Psi + |\Psi|^2 \nabla \cdot \mathbf{A}) \mathbf{r} d^3 \mathbf{r} = \tag{25}$$

$$\frac{q}{m} \int \left(\frac{\partial (\Psi^* \Psi A_x)}{\partial x} + \frac{\partial (\Psi^* \Psi A_y)}{\partial y} + \frac{\partial (\Psi^* \Psi A_z)}{\partial z} \right) \mathbf{r} d^3 \mathbf{r} = -\frac{q}{m} \langle \mathbf{A} \rangle \tag{26}$$

Combining these results gives the final answer

$$\langle \mathbf{v} \rangle = \frac{d \langle \mathbf{r} \rangle}{dt} = \frac{1}{m} \langle \mathbf{p} - q \mathbf{A} \rangle \tag{27}$$

We can now go one step further and work out $m \frac{d\langle \mathbf{v} \rangle}{dt}$, which will give us the force. We can use the same techniques as above.

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \frac{d\langle\mathbf{p}\rangle}{dt} - q\frac{d\langle\mathbf{A}\rangle}{dt} \tag{28}$$

$$= \frac{d}{dt}\int \Psi^* \frac{\hbar}{i}\nabla\Psi d^3\mathbf{r} - q\frac{d}{dt}\int \Psi^*\mathbf{A}\Psi d^3\mathbf{r} \tag{29}$$

$$= \frac{i}{\hbar}\int \left(\frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right)\Psi^*\frac{\hbar}{i}\nabla\Psi d^3\mathbf{r} + \tag{30}$$

$$\int \Psi^*\nabla\left(\left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right)\Psi\right) d^3\mathbf{r} - \tag{31}$$

$$q\frac{i}{\hbar}\int \left(\frac{1}{2m}\left(\frac{\hbar}{-i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right)\Psi^*\mathbf{A}\Psi d^3\mathbf{r} - q\left\langle\frac{\partial\mathbf{A}}{\partial t}\right\rangle + \tag{32}$$

$$q\frac{i}{\hbar}\int \left(\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\varphi\right)\Psi\mathbf{A}\Psi^* d^3\mathbf{r} \tag{33}$$

Here we can use the equations above for the wave function and its derivative to evaluate the operators:

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = \int\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi^* + \frac{\hbar q}{i}(2\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi^*\right) + q\varphi\Psi^*\right]\nabla\Psi d^3\mathbf{r} -$$
$$\int\Psi^*\nabla\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi + \frac{\hbar q}{-i}(2\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi\right) + q\varphi\Psi\right] d^3\mathbf{r} - \tag{34}$$

$$q\frac{i}{\hbar}\int\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi^* + \frac{\hbar q}{i}(2\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi^*\right) + q\varphi\Psi^*\right]\mathbf{A}\Psi d^3\mathbf{r} -$$
$$q\left\langle\frac{\partial\mathbf{A}}{\partial t}\right\rangle +$$
$$q\frac{i}{\hbar}\int\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi + \frac{\hbar q}{-i}(2\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi\right) + q\varphi\Psi\right]\mathbf{A}\Psi^* d^3\mathbf{r} \tag{35}$$

We can begin the process of simplifying this expression by looking first at the terms involving φ in the first and second lines. Working out the derivatives we see that these two terms combine to give

$$-q\int\Psi^*(\nabla\varphi)\Psi d^3\mathbf{r} = -q\langle\nabla\varphi\rangle \tag{36}$$

Combining this with the fourth line and using $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ we get the term $q\langle\mathbf{E}\rangle$.

Next, we see that all the terms in lines 3 and 5 that do *not* involve derivatives of the wave function cancel out in the subtraction.

Also, by looking at the terms involving $\nabla^2\Psi$ or $\nabla^2\Psi^*$ in lines 1 and 2, we can use integration by parts several times to show that these terms cancel out. We are therefore left with

$$\begin{aligned} m\frac{d\langle \mathbf{v}\rangle}{dt} = q\langle \mathbf{E}\rangle + \\ &\int \left[\frac{1}{2m}\left(\frac{\hbar q}{i}(2\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi^*\right)\right]\nabla\Psi d^3\mathbf{r} - \\ &\int \Psi^*\nabla\left[\frac{1}{2m}\left(\frac{\hbar q}{-i}(2\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot\mathbf{A}) + q^2|\mathbf{A}|^2\Psi\right)\right] d^3\mathbf{r} - \\ &q\frac{i}{\hbar}\int\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi^* + \frac{\hbar q}{i}(2\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\nabla\cdot\mathbf{A})\right)\right]\mathbf{A}\Psi d^3\mathbf{r} + \\ &q\frac{i}{\hbar}\int\left[\frac{1}{2m}\left(-\hbar^2\nabla^2\Psi + \frac{\hbar q}{-i}(2\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot\mathbf{A})\right)\right]\mathbf{A}\Psi^* d^3\mathbf{r} \end{aligned} \tag{37}$$

This expression splits into two parts: one with a factor of q and the other with a factor of q^2.

Consider the part with factor q^2.

$$\frac{q^2}{2m}\int[|\mathbf{A}|^2\Psi^*\nabla\Psi - \Psi^*\nabla(|\mathbf{A}|^2\Psi) - 2\Psi(\mathbf{A}\cdot\nabla\Psi^*)\mathbf{A} - \tag{38}$$

$$2\Psi^*(\mathbf{A}\cdot\nabla\Psi)\mathbf{A} - 2\Psi^*(\nabla\cdot\mathbf{A})\mathbf{A}]d^3\mathbf{r} \tag{39}$$

Since this is a vector integrand, it is easiest to evaluate if we consider each component of the vector separately. Taking the x component, we get

$$\begin{aligned} &\frac{q^2}{2m}\int\left[|\mathbf{A}|^2\Psi^*\frac{\partial\Psi}{\partial x} - 2\Psi^*\left(A_x\frac{\partial A_x}{\partial x} + A_y\frac{\partial A_y}{\partial x} + A_z\frac{\partial A_z}{\partial x}\right) - |\mathbf{A}|^2\Psi^*\frac{\partial\Psi}{\partial x}\right. \\ &\quad - 2A_x\Psi\left(A_x\frac{\partial\Psi^*}{\partial x} + A_y\frac{\partial\Psi^*}{\partial y} + A_z\frac{\partial\Psi^*}{\partial z}\right) - 2A_x\Psi^*\left(A_x\frac{\partial\Psi}{\partial x} + A_y\frac{\partial\Psi}{\partial y} + A_z\frac{\partial\Psi}{\partial z}\right) \\ &\quad \left. - 2\Psi\Psi^* A_x\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial x} + \frac{\partial A_z}{\partial x}\right)\right]d^3\mathbf{r} \end{aligned} \tag{40}$$

The first and last terms in the first line cancel, but no other cancellations are obvious at this stage. To proceed, we can use the integration by parts method again. Imposing the usual condition that the wave function must be zero at all infinite distances, we can integrate each bit of the first term in the second line as follows. In each case, the integrated part is omitted as it is zero in all cases, and the integration by parts is done with respect to the first integration variable.

$$-\frac{q^2}{m}\int \mathbf{\Psi} A_x^2 \frac{\partial \mathbf{\Psi}^*}{\partial x} dxdydz = \frac{q^2}{m}\int \mathbf{\Psi}^* \left(2A_x \frac{\partial A_x}{\partial x}\mathbf{\Psi} + A_x^2 \frac{\partial \mathbf{\Psi}}{\partial x}\right) dxdydz \tag{41}$$

$$-\frac{q^2}{m}\int \mathbf{\Psi} A_x A_y \frac{\partial \mathbf{\Psi}^*}{\partial y} dydxdz = \frac{q^2}{m}\int \mathbf{\Psi}^* \left(A_y \frac{\partial A_x}{\partial y}\mathbf{\Psi} + A_x \frac{\partial A_y}{\partial y}\mathbf{\Psi} + A_x A_y \frac{\partial \mathbf{\Psi}}{\partial y}\right) dydxdz \tag{42}$$

$$-\frac{q^2}{m}\int \mathbf{\Psi} A_x A_z \frac{\partial \mathbf{\Psi}^*}{\partial z} dzdxdy = \frac{q^2}{m}\int \mathbf{\Psi}^* \left(A_z \frac{\partial A_x}{\partial z}\mathbf{\Psi} + A_x \frac{\partial A_z}{\partial z}\mathbf{\Psi} + A_x A_z \frac{\partial \mathbf{\Psi}}{\partial z}\right) dzdxdy \tag{43}$$

Inserting these integrations into the first term in the second line of the x component above, we now find that most of the terms do cancel with other terms in the expression, and we are left with (using $\mathbf{B} = \nabla \times \mathbf{A}$):

$$-\frac{q^2}{m}\left\langle A_y \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) - A_z \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\rangle = -\frac{q^2}{m}\langle \mathbf{A} \times \mathbf{B}\rangle_x \tag{44}$$

The other two components of the q^2 term work out to the other two components of $\langle \mathbf{A} \times \mathbf{B}\rangle$.

Finally, we consider the q term. This term gets fairly involved and includes second derivatives, so to simplify the notation, we will indicate a vector component by a subscript x, y or z and a derivative by a superscript. Thus $A_y^{xz} = \partial^2 A_y / dxdz$ and so on.

Collecting together the terms in q from above we get for the integrand:

$$\frac{\hbar}{i}\frac{q}{2m}((2\mathbf{A}\cdot\nabla\Psi^* + \Psi^*\nabla\cdot\mathbf{A})\nabla\Psi + \Psi^*\nabla(2\mathbf{A}\cdot\nabla\Psi + \Psi\nabla\cdot\mathbf{A}) \tag{45}$$

$$-(\nabla^2\Psi^*)\mathbf{A}\Psi + (\nabla^2\Psi)\mathbf{A}\Psi^*) \tag{46}$$

As before, since this is a vector expression, we will look at the x component, since the derivation for the other two components is the same. Expanding all the terms and isolating the x component, we get

$$2(A_x\Psi^{*x}+A_y\Psi^{*y}+A_z\Psi^{*z}) \tag{47}$$
$$+\Psi^*(A_x^x+A_y^y+A_z^z)\Psi^x \tag{48}$$
$$+2\Psi^*(A_x^x\Psi^x+A_x\Psi^{xx}+A_y^x\Psi^y+A_y\Psi^{xy}+A_z^x\Psi^z+A_z\Psi^{xz}) \tag{49}$$
$$+\Psi^*\Psi^x(A_x^x+A_y^y+A_z^z)+\Psi^*\Psi(A_x^{xx}+A_y^{xy}+A_z^{xz}) \tag{50}$$
$$-A_x\Psi(\Psi^{*xx}+\Psi^{*yy}+\Psi^{*zz}) \tag{51}$$
$$+A_x\Psi^*(\Psi^{xx}+\Psi^{yy}+\Psi^{zz}) \tag{52}$$

From here, we need to apply the integration by parts method, but the trick is knowing where to apply it. Since the momentum $\mathbf{p}=(\hbar/i)\nabla$ is one of the operators we hope to identify in the final answer, we expect that terms involving the derivatives of Ψ will be present, while those involving the derivatives of Ψ^* will not. We therefore apply integration by parts to get rid of all derivatives of Ψ^* in this expression, so we will apply it to the first and fifth lines, since these are the only two lines where derivatives of Ψ^* appear.

We need to apply integration by parts once to the first line, and we get

$$2\int(A_x\Psi^{*x}+A_y\Psi^{*y}+A_z\Psi^{*z})d^3\mathbf{r}= \tag{53}$$
$$-2\int\Psi^*(A_x^x\Psi^x+A_x\Psi^{xx}+A_y^y\Psi^x+A_y\Psi^{xy}+A_z^z\Psi^x+A_z\Psi^{xz})d^3\mathbf{r} \tag{54}$$

Since the derivatives in the fifth line are all second order, we need to apply integration by parts twice to eliminate them. We get

$$-\int A_x\Psi(\Psi^{*xx}+\Psi^{*yy}+\Psi^{*zz})d^3\mathbf{r}=-\int\Psi^*(A_x^{xx}\Psi+2A_x^x\Psi^x+A_x\Psi^{xx})d^3\mathbf{r} \tag{55}$$
$$-\int\Psi^*(A_x^{yy}\Psi+2A_x^y\Psi^y+A_x\Psi^{yy})d^3\mathbf{r} \tag{56}$$
$$-\int\Psi^*(A_x^{zz}\Psi+2A_x^z\Psi^z+A_x\Psi^{zz})d^3\mathbf{r} \tag{57}$$

Replacing the integrands in lines 1 and 5 with these expressions, we can now cancel off many of the terms. When we are done, we are left with

$$\frac{\hbar}{i}\frac{q}{2m}\int\Psi^*(A_y^{xy}\Psi+2A_y^x\Psi^y-A_x^{yy}\Psi-2A_x^y\Psi^y-A_x^{zz}\Psi-2A_x^z\Psi^z+A_z^{xz}\Psi+2A_z^x\Psi^z)d^3\mathbf{r} \tag{58}$$

To put this in simpler form, we need to work out $\mathbf{p}\times\mathbf{B}-\mathbf{B}\times\mathbf{p}$ (since that's the answer given in the book). Since the momentum operator takes the derivative of its operand, we need to give this operator a function on which to operate. We therefore need to work out $(\mathbf{p}\times\mathbf{B}-\mathbf{B}\times\mathbf{p})\Psi$. Using $\mathbf{p}=(\hbar/i)\nabla$ and $\mathbf{B}=\nabla\times\mathbf{A}$, we get for the x component:

$$\frac{i}{\hbar}(\mathbf{p}\times\mathbf{B})_x\Psi = \frac{\partial(B_z\Psi)}{\partial y}-\frac{\partial(B_y\Psi)}{\partial z} \tag{59}$$

$$= \frac{\partial}{\partial y}((A_y^x - A_x^y)\Psi) - \frac{\partial}{\partial z}((A_x^z - A_z^x)\Psi) \tag{60}$$

$$= A_y^{xy}\Psi + A_y^x\Psi^y - A_x^{yy}\Psi - A_x^y\Psi^y - A_x^{zz}\Psi - A_x^z\Psi^z + A_z^{xz}\Psi + A_z^x\Psi^z \tag{61}$$

$$-\frac{i}{\hbar}(\mathbf{B}\times\mathbf{p})_x\Psi = -B_z\frac{\partial\Psi}{\partial y}+B_y\frac{\partial\Psi}{\partial z} \tag{62}$$

$$= A_y^x\Psi^y - A_x^y\Psi^y - A_x^z\Psi^z + A_z^x\Psi^z \tag{63}$$

$$(\mathbf{p}\times\mathbf{B}-\mathbf{B}\times\mathbf{p})_x\Psi = \frac{\hbar}{i}(A_y^{xy}\Psi + 2A_y^x\Psi^y - A_x^{yy}\Psi - 2A_x^y\Psi^y - A_x^{zz}\Psi - 2A_x^z\Psi^z + A_z^{xz}\Psi + 2A_z^x\Psi^z) \tag{64}$$

Therefore the q term evaluates to

$$\frac{q}{2m}\langle\mathbf{p}\times\mathbf{B}-\mathbf{B}\times\mathbf{p}\rangle \tag{65}$$

Putting everything together we get

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = q\langle\mathbf{E}\rangle + \frac{q}{2m}\langle\mathbf{p}\times\mathbf{B}-\mathbf{B}\times\mathbf{p}\rangle - \frac{q^2}{m}\langle\mathbf{A}\times\mathbf{B}\rangle \tag{66}$$

This completes the proof.

If the electric and magnetic fields are uniform, then their spatial derivatives are all zero. The only effect this has is in the expression for $\frac{i}{\hbar}(\mathbf{p}\times\mathbf{B})_x\Psi$ above, where all second derivatives of $\mathbf{A}$ are zero. In this case, $\langle\mathbf{p}\times\mathbf{B}\rangle = \langle\mathbf{p}\rangle\times\mathbf{B} = -\mathbf{B}\times\langle\mathbf{p}\rangle$ so using the result 27:

$$m\frac{d\langle\mathbf{v}\rangle}{dt} = q\mathbf{E} + \frac{q}{m}\langle\mathbf{p}\rangle\times\mathbf{B} - \frac{q^2}{m}\langle\mathbf{A}\rangle\times\mathbf{B} \tag{67}$$

$$= q\mathbf{E} + q\langle\mathbf{v}\rangle\times\mathbf{B} \tag{68}$$

This is the classical force law expressed in terms of quantum mean values.