

ENERGY-TIME UNCERTAINTY PRINCIPLE - GAUSSIAN FREE PARTICLE

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As another example of the energy-time uncertainty relation, we can look again at the example of a travelling free particle with a Gaussian wave packet. We have already worked out most of what we need to test the uncertainty relation:

$$\langle x \rangle = \frac{l\hbar t}{m} \quad (1)$$

$$\langle x^2 \rangle = \frac{1 + (2a\hbar t/m)^2 + a(2\hbar l t/m)^2}{4a} \quad (2)$$

$$\langle p^2 \rangle = \hbar^2(a + l^2) \quad (3)$$

From this we get

$$\langle H \rangle = \frac{\langle p^2 \rangle}{2m} \quad (4)$$

$$= \frac{\hbar^2(a + l^2)}{2m} \quad (5)$$

We still need $\langle H^2 \rangle$. From our previous calculations, we have the wave function:

$$\Psi(x, t) = \left(\frac{2a}{\pi}\right)^{1/4} \frac{e^{(-ax^2 + i(lx - \hbar l^2 t/2m))/(1 + 2i\hbar a t/m)}}{\sqrt{1 + 2i\hbar a t/m}} \quad (6)$$

We can calculate $\langle H^2 \rangle$ by direct integration, using Maple:

$$\langle H^2 \rangle = \frac{\hbar^4}{4m^2} \int_{-\infty}^{\infty} \left| \frac{d^2\Psi(x, t)}{dx^2} \right|^2 dx \quad (7)$$

$$= \frac{\hbar^4}{4m^2} (3a^2 + 6al^2 + l^4) \quad (8)$$

As a check on this result, we can work out the units (always a good test to make sure you haven't dropped a factor somewhere). From the original wave function, since exponents must be dimensionless, we know that a has dimensions distance⁻² and l has distance⁻¹. Planck's constant has dimensions of energy \times time, so the expression above has overall units of energy⁴ \times time⁴ \times mass⁻² \times distance⁻⁴ = energy². (Recall kinetic energy is $mv^2/2$.) It's also worth noting that $\langle H^2 \rangle$ is independent of time.

From here, we can get σ_H^2 :

$$\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 \quad (9)$$

$$= \frac{a\hbar^4}{2m^2} (a + 2l^2) \quad (10)$$

We also have, from above

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad (11)$$

$$= \frac{m^2 + 4(at\hbar)^2}{4am^2} \quad (12)$$

We're trying to show that $\sigma_H \sigma_x \geq \frac{\hbar}{2} \frac{d\langle x \rangle}{dt} = l\hbar^2/2m$ from above. So we want

$$\sigma_H^2 \sigma_x^2 \geq \frac{l^2 \hbar^4}{4m^2} \quad (13)$$

$$\frac{a\hbar^4}{2m^2} (a + 2l^2) \frac{m^2 + 4(at\hbar)^2}{4am^2} \geq \frac{l^2 \hbar^4}{4m^2} \quad (14)$$

The minimum of the LHS occurs at $t = 0$ so if the inequality is true there, it is true always. In this case, it reduces to

$$a + 2l^2 \geq 2l^2 \quad (15)$$

$$a \geq 0 \quad (16)$$

This final condition is certainly true (it is required for the Gaussian wave form to converge at large x), so the uncertainty condition is verified.