ENERGY-TIME UNCERTAINTY PRINCIPLE - INFINITE SQUARE WELL

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As an example of the energy-time uncertainty relation, we can look again at the example of a particle in the infinite square well, which starts off in a combination of the two lowest states:

$$\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right] \tag{1}$$

We would like to verify by explicit calculation that

$$\sigma_H \sigma_x \ge \frac{\hbar}{2}\frac{d\langle x\rangle}{dt} \tag{2}$$

Referring back to our calculations on that example, we can get some of the needed quantities immediately:

$$\frac{d\langle x\rangle}{dt} = \frac{8\hbar}{3ma}\sin(3\omega t) \tag{3}$$

$$\langle H\rangle = \frac{5\pi^2\hbar^2}{4ma^2} \tag{4}$$

$$\langle x\rangle = \frac{a}{2} - \frac{16a}{9\pi^2}\cos(3\omega t) \tag{5}$$

$$\Psi(x,t) = \frac{\sqrt{2}}{2}\psi_1(x)e^{-i\omega t} + \frac{\sqrt{2}}{2}\psi_2(x)e^{-4i\omega t} \tag{6}$$

$$E_i = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{7}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \tag{8}$$

We can calculate σ_H^2 using these equations:

$$(\hat{H}-\langle H\rangle)\Psi = \frac{\sqrt{2}}{2}[(E_1-\langle H\rangle])\psi_1 e^{-i\omega t} + (E_2-\langle H\rangle)\psi_2 e^{-4i\omega t}] \tag{9}$$

$$\sigma_H^2 = \langle(\hat{H}-\langle H\rangle)\Psi|(\hat{H}-\langle H\rangle)\Psi\rangle \tag{10}$$

$$= \frac{1}{2}[(E_1-\langle H\rangle])^2 + (E_2-\langle H\rangle)^2] \tag{11}$$

$$= \frac{9}{16}\left(\frac{\pi^2\hbar^2}{ma^2}\right)^2 \tag{12}$$

Thus

$$\sigma_H = \frac{3}{4}\frac{\pi^2\hbar^2}{ma^2} \tag{13}$$

To get σ_x, we can use the fact that, for any quantity A, $\sigma_A^2 = \langle A^2\rangle - \langle A\rangle^2$. We already know $\langle x\rangle$, so we can calculate $\langle x^2\rangle$ using integration. We get:

$$\langle x^2\rangle = \frac{1}{2}\int_0^a (x^2\psi_1^2(x) + x^2\psi_2^2(x) + 2x^2\psi_1\psi_2\cos(3\omega t))dx \tag{14}$$

$$= \frac{a^2}{144\pi^2}(48\pi^2 - 45 - 256\cos(3\omega t)) \tag{15}$$

where we did the integral using Maple. We now get σ_x (after simplifying):

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 \tag{16}$$

$$= \frac{a^2}{1296\pi^4}(108\pi^4 - 405\pi^2 - 4096\cos^2(3\omega t)) \tag{17}$$

$$= a^2\left[\frac{1}{12} - \frac{5}{16\pi^2} - \left(\frac{16}{9\pi^2}\cos 3\omega t\right)^2\right] \tag{18}$$

If we define

$$\beta^2 \equiv 108\pi^4 - 405\pi^2 - 4096\cos^2(3\omega t) \tag{19}$$

then

$$\sigma_x = \frac{a}{36\pi^2}\beta \tag{20}$$

and

$$\sigma_H\sigma_x = \frac{1}{48}\frac{\hbar^2}{ma}\beta \tag{21}$$

We would like to show that this satisfies the inequality

$$\frac{1}{48}\frac{\hbar^2}{ma}\beta \geq \frac{\hbar}{2}\frac{d\langle x\rangle}{dt} = \frac{4\hbar^2}{3ma}\sin(3\omega t) \tag{22}$$

which will be true if $\beta \geq 64\sin(3\omega t)$, or $\beta^2 \geq 4096\sin^2(3\omega t)$. Substituting for β from 19 we get

$$108\pi^4 - 405\pi^2 - 4096\cos^2(3\omega t) \stackrel{?}{\geq} 4096\sin^2(3\omega t) \tag{23}$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096\cos^2(3\omega t) + 4096\sin^2(3\omega t) \tag{24}$$

$$108\pi^4 - 405\pi^2 \stackrel{?}{\geq} 4096 \tag{25}$$

Putting in the numbers on the LHS leads to 6522.992 > 4096, which is true so the uncertainty principle is satisfied for this case.