FIRST BORN APPROXIMATION - SOFT-SPHERE SCATTERING

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The first Born approximation for the scattering amplitude comes from the integral form of the Schrödinger equation:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0 \tag{1}$$

This equation is valid for all $\mathbf{r}$, even for positions near to the origin where the scattering potential V could be significantly different from zero. In a scattering problem, the detector is usually situated far from the scattering region, so for all $\mathbf{r}$ of interest, $r \gg r_0$ and we're well outside the region where $V \neq 0$. In such cases, we can approximate (see Griffiths, section 11.4.2 for details) the integral equation by

$$\psi(\mathbf{r}) \cong Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0 \tag{2}$$

$$= A\left[e^{ikz} - \frac{m}{2\pi\hbar^2 A}\frac{e^{ikr}}{r}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0\right] \tag{3}$$

where $\mathbf{k} \equiv k\hat{\mathbf{r}}$ is a vector pointing from the origin to the detector (that is, parallel to $\mathbf{r}$). The first term on the RHS represents the incoming plane wave, as usual.

Since the scattering amplitude f is the coefficient of e^{ikr}/r inside the square brackets, we have

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\mathbf{k}\cdot\mathbf{r}_0} V(\mathbf{r}_0)\psi(\mathbf{r}_0)\,d^3\mathbf{r}_0 \tag{4}$$

This formula still doesn't help us much, since we still need to know the wave function ψ inside the scattering region where $V \neq 0$. The Born approximation assumes that the potential is weak, so that the incoming plane wave Ae^{ikz} doesn't change much after it scatters. That is, we assume that, for all points where the integrand is non-zero:

$$\psi(\mathbf{r}_0) \approx \psi_0(\mathbf{r}_0) \tag{5}$$

The incident plane wave has a wave vector of magnitude k that is parallel to $\hat{\mathbf{z}}$, which we can write as

$$\mathbf{k}' \equiv k\hat{\mathbf{z}} \tag{6}$$

For some position $\mathbf{r}_0$ with z component z_0:

$$kz_0 = \mathbf{k}' \cdot \mathbf{r}_0 \tag{7}$$

so the assumption above amounts to saying that

$$\psi(\mathbf{r}_0) \approx Ae^{i\mathbf{k}'\cdot\mathbf{r}_0} \tag{8}$$

This assumption gives us an approximation for f:

$$f(\theta,\phi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0} V(\mathbf{r}_0)\, d^3\mathbf{r}_0 \tag{9}$$

It's important to remember that, from the point of view of the integral, $\mathbf{k}$ and $\mathbf{k}'$ are constant, with the direction of $\mathbf{k} = k\hat{\mathbf{r}}$ being how the polar angles θ and ϕ are specified. The vector $\mathbf{k}' = k\hat{\mathbf{z}}$ is always the same as it specifies the direction of the incident particle.

For a spherically symmetric potential, the integral can be simplified a bit by defining the vector

$$\boldsymbol{\kappa} \equiv \mathbf{k}' - \mathbf{k} \tag{10}$$

The vector is the base of an isosceles triangle with sides $\mathbf{k}'$ and $\mathbf{k}$, so since the angle between $\mathbf{k}'$ and $\mathbf{k}$ is θ ($\mathbf{k}'$ is the direction to the detector and $\mathbf{k}$ is the direction of the incident particle, so θ is the scattering angle), we can divide the isosceles triangle into two symmetric right angled triangles by drawing a line from the origin to the the midpoint of $\boldsymbol{\kappa}$. The length of the base is $\kappa/2$ which is also $k\sin\frac{\theta}{2}$, so

$$\kappa = 2k\sin\frac{\theta}{2} \tag{11}$$

Letting the polar axis in the integral 9 lie along $\boldsymbol{\kappa}$ we get $(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}_0 = \kappa r_0\cos\theta_0$ [where θ_0 is the polar angle of integration, *not* θ!] and

$$f(\theta) \approx -\frac{m}{2\pi\hbar^2}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{i\kappa r_0\cos\theta_0} V(r_0)\, r_0^2 \sin\theta_0 d\phi_0 d\theta_0 dr_0 \tag{12}$$

$$= -\frac{2m}{\hbar^2\kappa}\int_0^\infty V(r_0)\, r_0 \sin(\kappa r_0)\, dr_0 \tag{13}$$

Example 1. Soft-sphere scattering. A *soft sphere* is defined by the potential

$$V(\mathbf{r})=\begin{cases}V_0 & r\le a\\ 0 & r>a\end{cases} \tag{14}$$

where $V_0>0$ is a constant. [The hard sphere takes $V_0=\infty$.] From 13, we can get the Born approximation for the scattering amplitude:

$$f(\theta)\approx-\frac{2mV_0}{\hbar^2\kappa}\int_0^a r\sin(\kappa r)\,dr \tag{15}$$

$$=-\frac{2mV_0}{\hbar^2\kappa^3}\left[\sin(\kappa a)-a\kappa\cos(\kappa a)\right] \tag{16}$$

with the θ dependence given by the definition of κ in 11.

For low energy scattering $\kappa a\ll 1$ and we can expand the sin and cos.

$$\sin(\kappa a)-a\kappa\cos(\kappa a)=\kappa a-\frac{(\kappa a)^3}{3!}+\ldots-\kappa a\left(1-\frac{(\kappa a)^2}{2!}+\ldots\right) \tag{17}$$

$$=\frac{(\kappa a)^3}{3}+\ldots \tag{18}$$

To this order, the scattering amplitude is

$$f(\theta)\approx-\frac{2mV_0a^3}{3\hbar^3} \tag{19}$$

which agrees with equation 11.82 in Griffiths.

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