FREE PARTICLE - TRAVELLING WAVE PACKET

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We've looked at the stationary Gaussian wave packet for the free particle. The initial wave function in that case was

$$\Psi(x,0)=Ae^{-ax^{2}} \tag{1}$$

We can turn this into a travelling Gaussian wave by adding a factor to the wave function:

$$\Psi(x,0)=Ae^{-ax^{2}}e^{ilx} \tag{2}$$

where l is a real constant.

Since we have added only a complex exponential, the normalization condition is the same as for the stationary case:

$$A=\left(\frac{2a}{\pi}\right)^{1/4} \tag{3}$$

To find $\Psi(x,t)$ we follow the same procedure as in the stationary case. So we get

$$\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(k)e^{ikx}e^{-i\hbar k^{2}t/2m}dk \tag{4}$$

Given the initial wave function, we can find $\phi(k)$ via Plancherel's theorem:

$$\begin{aligned}\phi(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Psi(x,0)e^{-ikx}dx &(5)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{2a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty}e^{-ax^{2}-ikx+ilx}dx &(6)\\ &= \left(\frac{1}{2\pi a}\right)^{1/4}e^{-(k-l)^{2}/4a} &(7)\end{aligned}$$

So we can now find the general solution:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i(kx-\hbar k^2 t/2m)}dk \tag{8}$$

$$= \left(\frac{2a}{\pi}\right)^{1/4}\frac{e^{\left(-ax^2+i(lx-\hbar l^2 t/2m)\right)/(1+2i\hbar at/m)}}{\sqrt{1+2i\hbar at/m}} \tag{9}$$

where Maple was used for the integral.

Calculating $|\Psi(x,t)|^2$ can be done using Maple, with the result:

$$|\Psi(x,t)|^2 = \sqrt{\frac{2}{\pi}}we^{-2w^2(\hbar lt/m-x)^2} \tag{10}$$

with

$$w \equiv \left(\frac{a}{1+(2\hbar at/m)^2}\right)^{1/2} \tag{11}$$

The results above reduce to the stationary wave packet when $l=0$.

At $t=0$, $w=\sqrt{a}$, so $|\Psi(x,0)|^2 = \sqrt{2a/\pi}e^{-2ax^2}$ which is correct. The wave packet at $t=0$ is therefore a Gaussian centred at $x=0$. As t increases, w gets smaller but in this case, the peak of the Gaussian moves according to $x_{peak} = \hbar lt/m$. The speed of the peak is $x/t = \hbar l/m$.

By direct integration we find, $\langle x\rangle = \hbar lt/m$. Calculating the other means requires a bit of effort but we can use Maple to do most of it. The results are:

$$\langle p\rangle = l\hbar \tag{12}$$

$$\langle x^2\rangle = \frac{1+(2a\hbar t/m)^2+a(2\hbar lt/m)^2}{4a} \tag{13}$$

$$\langle p^2\rangle = \hbar^2(a+l^2) \tag{14}$$

All these results reduce to those for the stationary wave packet when $l=0$.

The uncertainty principle thus becomes

$$\sigma_x\sigma_p = \sqrt{(\langle x^2\rangle-\langle x\rangle^2)(\langle p^2\rangle-\langle p\rangle^2)} \tag{15}$$

$$= \frac{\hbar}{2}\sqrt{1+(2\hbar at/m)^2} \tag{16}$$

which is the same result as in the stationary wave packet. Thus although the packet here travels with a constant speed, it spreads out at the same rate as the stationary packet.

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