FREE PARTICLE IN MOMENTUM SPACE

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Post date: 29 July 2021.

Since the Hamiltonian for a free particle is $H = p^2/2m$, the Schrodinger equation in momentum space is

$$i\hbar\frac{\partial\Phi}{\partial t} = \frac{p^2}{2m}\Phi \tag{1}$$

so the solution can be found by simply integrating with respect to t:

$$\Phi(p,t) = e^{-ip^2t/2m\hbar}\Phi(p,0) \tag{2}$$

We looked at the travelling Gaussian wave packet in free space earlier. Its initial state in position space is

$$\Psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}e^{ilx} \tag{3}$$

To find $\Phi(p,0)$ we use the conversion to momentum space we found earlier:

$$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}}\left(\frac{2a}{\pi}\right)^{1/4}\int_{-\infty}^{\infty} e^{-ax^2+ilx-ipx/\hbar}dx \tag{4}$$

From the analysis of the travelling Gaussian packet we see that the integral is the same as that done when calculating $\phi(k)$ if we replace k with $p/\hbar$. Therefore

$$\Phi(p,0) = \left(\frac{2}{\pi a}\right)^{1/4}\frac{1}{\sqrt{2\hbar}}e^{-(p/\hbar-l)^2/4a} \tag{5}$$

Using 2, we have the full solution for $\Phi(p,t)$:

$$\Phi(p,t) = \left(\frac{2}{\pi a}\right)^{1/4}\frac{1}{\sqrt{2\hbar}}e^{-ip^2t/2m\hbar}e^{-(p/\hbar-l)^2/4a} \tag{6}$$

Also

$$|\Phi(p,t)|^2 = \frac{1}{\hbar}\frac{1}{\sqrt{2\pi a}}e^{-(p/\hbar-l)^2/2a} \tag{7}$$

which is independent of time. (As a check, we can integrate this over all p and verify that this integral is 1.)

We can calculate the means for momentum in the usual way:

$$\langle p\rangle = \frac{1}{\hbar}\frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} p e^{-(p/\hbar - l)^2/2a}dp \tag{8}$$

$$= \hbar l \tag{9}$$

$$\langle p^2\rangle = \frac{1}{\hbar}\frac{1}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} p^2 e^{-(p/\hbar - l)^2/2a}dp \tag{10}$$

$$= \hbar^2(l^2 + a) \tag{11}$$

Both results agree with those in the analysis of the travelling Gaussian packet.

For the mean energy, we have

$$\langle H\rangle = \left\langle \frac{p^2}{2m}\right\rangle \tag{12}$$

$$= \frac{\hbar^2}{2m}(l^2 + a) \tag{13}$$

$$= \frac{\langle p\rangle^2}{2m} + \frac{a\hbar^2}{2m} \tag{14}$$

Referring back to the stationary Gaussian wave packet in free space, we see that $\langle p^2\rangle = a\hbar^2$, so the energy is the sum of that for a stationary Gaussian wave packet and the term $\langle p\rangle^2/2m$. For the travelling packet, there is a net non-zero average momentum, so $\langle p\rangle$ is non-zero. Thus the energy arises from the inherent energy of the wave packet, plus the kinetic energy of motion of the packet.