FREE PARTICLE PROPAGATOR

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The Schrödinger equation was justified by the assumption that a free particle behaves like a wave, and thus can be expressed by a function of the form

$$\psi(x,t) = e^{i(px-Et)\hbar} \tag{1}$$

$$= e^{i\left(kx-\hbar k^2 t/2m\right)} \tag{2}$$

where we've used the de Broglie equation for the particle's momentum

$$p = \hbar k \tag{3}$$

and the energy is therefore

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{4}$$

In practice, a particle is represented by a superposition of many such waves with different momenta and energies. We can therefore represent a particle by the integral

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int dk\, \Phi(k)\, e^{i\left(kx-\hbar k^2 t/2m\right)} \tag{5}$$

The function $\Phi(k)$ is a weighting function which tells us how much the wave with wave vector k contributes to the particle.

If we know the initial state $\psi(x,0)$ we can work out $\Phi(k)$ by using a Fourier transform. We have

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int dk\, \Phi(k)\, e^{ikx} \tag{6}$$

The Fourier transform (actually an inverse transform) is

$$\frac{1}{\sqrt{2\pi}}\int dx\, e^{-ik'x}\psi(x,0) = \frac{1}{2\pi}\int dx\, e^{-ik'x}\int dk\,\Phi(k)\,e^{ikx} \tag{7}$$

$$= \frac{1}{2\pi}\int dk\,\Phi(k)\int dx\, e^{i(k-k')x} \tag{8}$$

$$= \int dk\,\Phi(k)\,\delta\left(k-k'\right) \tag{9}$$

$$= \Phi\left(k'\right) \tag{10}$$

where we used the integral representation of the delta function. Thus we can determine the weighting function as a Fourier transform of the initial conditions:

$$\Phi(k) = \frac{1}{\sqrt{2\pi}}\int dx\, e^{-ikx}\psi(x,0) \tag{11}$$

Using 5, we can eliminate $\Phi(k)$ and write the general time-dependent wave directly in terms of the initial state.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\,\frac{1}{\sqrt{2\pi}}\int dx'\, e^{-ikx'}\psi\left(x',0\right)e^{i\left(kx-\hbar k^2 t/2m\right)} \tag{12}$$

$$= \frac{1}{2\pi}\int dx'\,\left[\int dk\, e^{i\left[k(x-x')-\hbar k^2 t/2m\right]}\right]\psi\left(x',0\right) \tag{13}$$

Although we can't do the integral over x' without knowing the initial state $\psi(x',0)$, we *can* do the integral over k, since all functions that depend on k are fully specified. I used Maple to do the integral, with the result

$$\frac{1}{2\pi}\int dk\, e^{i\left[k(x-x')-\hbar k^2 t/2m\right]} = \sqrt{\frac{m}{2\pi i\hbar t}}e^{i(x-x')^2 m/2\hbar t} \tag{14}$$

If we started with a non-zero initial time t_0 then we begin with

$$\psi(x,t_0) = \frac{1}{\sqrt{2\pi}}\int dk\,\Phi(k)\,e^{i\left(kx-\hbar k^2 t_0/2m\right)} \tag{15}$$

Tracing through the derivation yields, instead of 10

$$\frac{1}{\sqrt{2\pi}}\int dx\, e^{-ik'x}\psi(x,t_0) = \Phi\left(k'\right)e^{-i\hbar k'^2 t_0/2m} \tag{16}$$

$$\Phi(k) = \frac{1}{\sqrt{2\pi}}e^{i\hbar k^2 t_0/2m}\int dx\, e^{-ikx}\psi(x,t_0) \tag{17}$$

Continuing, we have

$$\psi(x,t,t_0) = \frac{1}{\sqrt{2\pi}} \int dk\, \frac{1}{\sqrt{2\pi}} \int dx'\, e^{-ikx'} \psi\left(x', t_0\right) e^{i\left(kx - \hbar k^2 (t-t_0)/2m\right)} \tag{18}$$

$$= \frac{1}{2\pi} \int dx' \left[\int dk\, e^{i\left[k(x-x') - \hbar k^2 (t-t_0)/2m\right]}\right] \psi\left(x', t_0\right) \tag{19}$$

with the integral becoming

$$\frac{1}{2\pi} \int dk\, e^{i\left[k(x-x') - \hbar k^2 (t-t_0)/2m\right]} = \sqrt{\frac{m}{2\pi i \hbar (t-t_0)}} e^{i(x-x')^2 m/2\hbar(t-t_0)} \tag{20}$$

Thus we see that $\psi(x,t)$ depends only on the time *difference* from the initial state, and not on the absolute time. As such, we can view the quantity in the square brackets in 19 as a *propagator*, as it describes how the wave function propagates from the initial time t_0 up to the present time t. Shankar calls the propagator U and Berman calls it K, but they are the same thing. Using Shankar's notation, we then have

$$\psi(x,t) = \int dx'\, U\left(x, x', t - t_0\right) \psi\left(x', t_0\right) \tag{21}$$

with

$$U\left(x, x', t - t_0\right) = \frac{1}{2\pi} \int dk\, e^{ik(x-x') - \hbar k^2 (t-t_0)/2m} \tag{22}$$

$$= \sqrt{\frac{m}{2\pi i \hbar (t-t_0)}} e^{i(x-x')^2 m/2\hbar(t-t_0)} \tag{23}$$

REFERENCES

(1) Shankar, R. (1994), *Principles of Quantum Mechanics*, Plenum Press, Chapter 5.
(2) Berman, Paul R. (2018), *Introductory Quantum Mechanics*, Springer, Chapter 3.

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