GEOMETRIC PHASE IS ALWAYS ZERO FOR REAL WAVE FUNCTIONS

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Here's another example of calculating phases in the adiabatic theorem which says that if a system starts out in the nth state of a time-dependent hamiltonian, and the hamiltonian changes slowly compared to the internal period of the time-independent wave function (that is, the time scale over which the hamiltonian changes is much longer than $\hbar/E_n$), then after a time t the system will end up in state

$$\Psi_n(t)=e^{i\theta_n(t)}e^{i\gamma_n(t)}\psi_n(t) \tag{1}$$

where

$$\theta_n(t)\equiv-\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \tag{2}$$

$$\gamma_n(t)\equiv i\int_0^t\left\langle\psi_n(t')\left|\frac{\partial}{\partial t'}\psi_n(t')\right.\right\rangle dt' \tag{3}$$

θ is called the *dynamic phase* and γ is called the *geometric phase*. If ψ_n is real, then γ_n is always zero, as we can see by differentiating the normalization condition:

$$\langle\psi_n|\psi_n\rangle=1 \tag{4}$$

$$\frac{d}{dt}\langle\psi_n|\psi_n\rangle=0 \tag{5}$$

$$=\left\langle\dot{\psi}_n|\psi_n\right\rangle+\left\langle\psi_n|\dot{\psi}_n\right\rangle \tag{6}$$

$$=\left\langle\psi_n|\dot{\psi}_n\right\rangle^*+\left\langle\psi_n|\dot{\psi}_n\right\rangle \tag{7}$$

$$=2\Re\left(\left\langle\psi_n|\dot{\psi}_n\right\rangle\right) \tag{8}$$

That is, $\left\langle\psi_n(t')\left|\frac{\partial}{\partial t'}\psi_n(t')\right.\right\rangle$ must be purely imaginary, so if ψ_n is real, the bracket must be zero. This also means that γ is always real.

We can multiply the real wave function ψ_n by a phase factor $e^{i\phi_n}$ where ϕ_n is a real function of whatever parameters are dependent on time in the hamiltonian (but ϕ_n is not a function of x). In that case we have a new wave function (we'll drop the subscript n to save time):

$$\psi' = e^{i\phi}\psi \tag{9}$$

$$\dot{\psi}' = i\dot{\phi}e^{i\phi}\psi + e^{i\phi}\dot{\psi} \tag{10}$$

$$\left\langle \psi' \middle| \dot{\psi}' \right\rangle = \left\langle \psi \middle| i\dot{\phi}\psi + \dot{\psi} \right\rangle \tag{11}$$

$$= i\left\langle \psi \middle| \dot{\phi}\psi \right\rangle + \left\langle \psi \middle| \dot{\psi} \right\rangle \tag{12}$$

$$= i\dot{\phi} \tag{13}$$

where in the last line we took $\dot{\phi}$ outside the bracket since it doesn't depend on x and used $\left\langle \psi \middle| \dot{\psi} \right\rangle = 0$. The geometric phase for the modified wave function is therefore

$$\gamma' = i\int_0^t i\dot{\phi}dt' \tag{14}$$

$$= -(\phi(t) - \phi(0)) \tag{15}$$

Putting this back into 1 we get

$$\Psi'(t) = e^{i\theta(t)}e^{-i(\phi(t)-\phi(0))}\psi'(t) \tag{16}$$

$$= e^{i\theta(t)}e^{-i(\phi(t)-\phi(0))}e^{i\phi(t)}\psi(t) \tag{17}$$

$$= e^{i\theta(t)}e^{i\phi(0)}\psi(t) \tag{18}$$

Although the wave function picks up a constant phase $\phi(0)$, there is no time-dependent geometric phase.