# HADAMARD'S LEMMA

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We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega)=e^{i\Omega} \tag{1}$$

Here we'll look at a special function of two operators of the form

$$h(A,B)=e^{A}Be^{-A} \tag{2}$$

If $[A,B]=0$, we can cancel the two exponentials and get the result $h(A,B)=B$. However, if $[A,B]\neq 0$ the two exponentials must remain separated by the middle $B$ operator. To get a simpler form for this function, we'll consider the auxiliary function

$$f(t)=e^{tA}Be^{-tA} \tag{3}$$

where $t$ is some parameter. We'll need the first 3 derivatives at $t=0$:

$$f(0)=B \tag{4}$$
$$f'(t)=Ae^{tA}Be^{-tA}-e^{tA}Be^{-tA}A \tag{5}$$
$$=e^{tA}[A,B]e^{-tA} \tag{6}$$
$$f'(0)=[A,B] \tag{7}$$
$$f''(t)=Ae^{tA}[A,B]e^{-tA}-e^{tA}[A,B]e^{-tA}A \tag{8}$$
$$=e^{tA}[A,[A,B]]e^{-tA} \tag{9}$$
$$f''(0)=[A,[A,B]] \tag{10}$$
$$f'''(t)=e^{tA}[A,[A,[A,B]]]e^{-tA} \tag{11}$$
$$f'''(0)=[A,[A,[A,B]]] \tag{12}$$

We can now write a Taylor expansion of 3 around $t=0$:

$$e^{tA}Be^{-tA} = f(0)+tf'(0)+\frac{t^2}{2}f''(0)+\frac{t^3}{6}f'''(0)+\ldots \tag{13}$$

$$= B+[A,B]t+[A,[A,B]]\frac{t^2}{2!}+[A,[A,[A,B]]]\frac{t^3}{3!}+\ldots \tag{14}$$

Taking $t=1$ gives the required expansion

$$e^{A}Be^{-A} = B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\ldots \tag{15}$$

This is known as *Hadamard's lemma*.

If we introduce the notation

$$\mathrm{ad}_A(B) \equiv [A,B] \tag{16}$$

$$\mathrm{ad}_A\mathrm{ad}_A(B) = [A,[A,B]] \tag{17}$$

and in general $(\mathrm{ad}_A)^n(B)$ is the $n$th order commutator of $A$ with $B$, then we can write Hadamard's lemma as

$$e^{A}Be^{-A} = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{ad}_A)^n(B) \tag{18}$$

$$= \exp(\mathrm{ad}_A)(B) \tag{19}$$

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