

## HADAMARD'S LEMMA

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We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega) = e^{i\Omega} \quad (1)$$

Here we'll look at a special function of two operators of the form

$$h(A, B) = e^A B e^{-A} \quad (2)$$

If  $[A, B] = 0$ , we can cancel the two exponentials and get the result  $h(A, B) = B$ . However, if  $[A, B] \neq 0$  the two exponentials must remain separated by the middle  $B$  operator. To get a simpler form for this function, we'll consider the auxiliary function

$$f(t) = e^{tA} B e^{-tA} \quad (3)$$

where  $t$  is some parameter. We'll need the first 3 derivatives at  $t = 0$ :

$$f(0) = B \quad (4)$$

$$f'(t) = A e^{tA} B e^{-tA} - e^{tA} B e^{-tA} A \quad (5)$$

$$= e^{tA} [A, B] e^{-tA} \quad (6)$$

$$f'(0) = [A, B] \quad (7)$$

$$f''(t) = A e^{tA} [A, B] e^{-tA} - e^{tA} [A, B] e^{-tA} A \quad (8)$$

$$= e^{tA} [A, [A, B]] e^{-tA} \quad (9)$$

$$f''(0) = [A, [A, B]] \quad (10)$$

$$f'''(t) = e^{tA} [A, [A, [A, B]]] e^{-tA} \quad (11)$$

$$f'''(0) = [A, [A, [A, B]]] \quad (12)$$

We can now write a Taylor expansion of 3 around  $t = 0$ :

$$e^{tA} B e^{-tA} = f(0) + t f'(0) + \frac{t^2}{2} f''(0) + \frac{t^3}{6} f'''(0) + \dots \quad (13)$$

$$= B + [A, B] t + [A, [A, B]] \frac{t^2}{2!} + [A, [A, [A, B]]] \frac{t^3}{3!} + \dots \quad (14)$$

Taking  $t = 1$  gives the required expansion

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots \quad (15)$$

This is known as *Hadamard's lemma*.

If we introduce the notation

$$\text{ad}_A(B) \equiv [A, B] \quad (16)$$

$$\text{ad}_A \text{ad}_A(B) = [A, [A, B]] \quad (17)$$

and in general  $(\text{ad}_A)^n(B)$  is the  $n$ th order commutator of  $A$  with  $B$ , then we can write Hadamard's lemma as

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_A)^n(B) \quad (18)$$

$$= \exp(\text{ad}_A)(B) \quad (19)$$

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