

HARMONIC OSCILLATOR - EXAMPLE STARTING STATE

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Suppose a particle in the harmonic oscillator potential starts out in the state

$$\Psi(x, 0) = A \left(1 - 2\sqrt{\frac{m\omega}{\hbar}} x \right)^2 e^{-m\omega x^2/2\hbar} \quad (1)$$

We can find the expectation value of the energy by expressing the given wave function as a linear combination of Hermite polynomials, since these form the orthonormal basis of solutions in the harmonic oscillator potential. Using the definitions

$$\alpha \equiv \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \quad (2)$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}} x \quad (3)$$

we have

$$\Psi(x, 0) = A(1 - 2\xi)^2 e^{-\xi^2/2} \quad (4)$$

Normalizing $\Psi(x, 0)$, by hand or using Maple, we find

$$\int_{-\infty}^{\infty} |\Psi(\xi, 0)|^2 d\xi = 1 \quad (5)$$

$$|A|^2 = \frac{\alpha^2}{25} \quad (6)$$

The stationary states of the Harmonic oscillator are

$$\psi_n(\xi) = \alpha \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2} \quad (7)$$

where $H_n(\xi)$ is the n th Hermite polynomial. The first 3 Hermite polynomials are

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= 2\xi \\
H_2 &= 4\xi^2 - 2
\end{aligned} \tag{8}$$

Taking A to be real and positive, and expanding in terms of the Hermite polynomials, we have from 4 and 8

$$\Psi(x, 0) = \frac{\alpha}{5}(4\xi^2 - 4\xi + 1)e^{-\xi^2/2} \tag{9}$$

$$= \frac{\alpha}{5}(H_2 - 2H_1 + 3H_0)e^{-\xi^2/2} \tag{10}$$

$$= \frac{1}{5}(2\sqrt{2}\psi_2 - 2\sqrt{2}\psi_1 + 3\psi_0) \tag{11}$$

The expectation value for the energy can now be calculated from the energies of the lowest 3 states of the harmonic oscillator: $E_n = (n + \frac{1}{2})\hbar\omega$. We get:

$$\langle E \rangle = \frac{\hbar\omega}{25} \left(8 \times \frac{5}{2} + 8 \times \frac{3}{2} + 9 \times \frac{1}{2} \right) = \frac{73}{50}\hbar\omega \tag{12}$$

Now suppose that at some later time T the wave function has changed to

$$\Psi(x, T) = B \left(1 + 2\sqrt{\frac{m\omega}{\hbar}}x \right)^2 e^{-m\omega x^2/2\hbar} \tag{13}$$

$$\Psi(\xi, T) = B(1 + 2\xi)^2 e^{-\xi^2/2} \tag{14}$$

(The only changes from $\Psi(x, 0)$ are the plus sign in the middle factor and the different normalization constant B .)

By normalizing the new wave function, we find that $|B|^2 = \frac{\alpha^2}{25} = |A|^2$. This time, we can't assume that B is real, but it can differ from A by a complex factor q such that $|q|^2 = 1$, that is, by a phase. If we convert the given wave function into Hermite polynomials as above, we get:

$$\Psi(\xi, T) = q\frac{\alpha}{5}(H_2 + 2H_1 + 3H_0)e^{-\xi^2/2} \tag{15}$$

$$= \frac{q}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0) \tag{16}$$

Thus the only difference between this function and the initial function (apart from the factor q) is that the coefficient of ψ_1 has changed sign. Since the general time-dependent solution is given by (with the lower limit on the

sum changed to 0 because the ground state for the harmonic oscillator has index 0 rather than 1):

$$\Psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \quad (17)$$

we need to find the time T such that $-e^{-iE_1 T/\hbar} = e^{-iE_0 T/\hbar} = e^{-iE_2 T/\hbar}$. Substituting the energies, we find:

$$-e^{-3i\omega T/2} = e^{-i\omega T/2} = e^{-5i\omega T/2} \quad (18)$$

That is

$$\frac{3\omega T}{2} = \frac{\omega T}{2} + (2m+1)\pi \quad (19)$$

$$\frac{5\omega T}{2} = \frac{\omega T}{2} + 2r\pi \quad (20)$$

for some integers m and r . From the first equation, we get $\omega T = (2m+1)\pi$ and from the second, $2\omega T = 2r\pi$. The smallest $T > 0$ occurs when $m = 0$ and $r = 1$, giving $T = \pi/\omega$. Then $e^{-3i\pi/2} = +i$, and $e^{-\pi i/2} = e^{-5\pi i/2} = -i$. By setting $q = -i$, we get the wave function at $t = T = \pi/\omega$:

$$\Psi(x, T) = -\frac{i}{5}(2\sqrt{2}\psi_2 + 2\sqrt{2}\psi_1 + 3\psi_0) \quad (21)$$