HARMONIC OSCILLATOR - POSITION, MOMENTUM AND ENERGY

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 18 May 2021.

The lowest two states for the harmonic oscillator are the ground state ψ_0 and first excited state ψ_1. We can work out some mean values of various quantities using explicit integration.

Consider first the ground state ψ_0. Using the substitutions $\xi \equiv \sqrt{m\omega/\hbar}x$ and $\alpha \equiv (m\omega/\pi\hbar)^{1/4}$ we have $\psi_0 = \alpha e^{-\xi^2/2}$. We can simplify the operations considerably if we note the even and odd natures of some of the functions to be integrated. Since $\psi_0(x)$ is even, $x\psi_0^2(x)$ is odd, so $\langle x\rangle = 0$. To calculate $\langle p\rangle$, since the operator $p = (\hbar/i)d/dx$, we need the derivative $d\psi_0/dx = \sqrt{\hbar/m\omega}d\psi_0/d\xi = \sqrt{\hbar/m\omega}(-2\xi)\psi_0$. This is again an odd function, so $\langle p\rangle = 0$ as well.

For the mean square values, we do need to do some integrals (I've used Maple for this).

$$\langle x^2\rangle = \frac{\hbar}{m\omega}\langle\xi^2\rangle \tag{1}$$

$$= \left(\frac{\hbar}{m\omega}\right)^{3/2}\alpha^2\int_{-\infty}^{\infty}\xi^2 e^{-\xi^2}d\xi \tag{2}$$

$$= \frac{\hbar}{2m\omega} \tag{3}$$

Here we used $d\xi \equiv \sqrt{m\omega/\hbar}\,dx$ to convert the integration variable from x to ξ in the second line.

$$\langle p^2\rangle = \sqrt{m\omega}\hbar^{3/2}\alpha^2\int_{-\infty}^{\infty}\left(1-\xi^2\right)e^{-\xi^2}d\xi \tag{4}$$

$$= \frac{1}{2}m\omega\hbar \tag{5}$$

For ψ_1 we have $\psi_1(\xi) = \sqrt{\frac{2\pi\hbar}{m\omega}}\alpha^3\xi e^{-\xi^2/2}$ which is an odd function. The square of an odd function is an even function, so $\psi_1^2(\xi)$ is even, which means that the function to be integrated to find $\langle x\rangle$ is again the product of an even function and an odd function, so $\langle x\rangle = 0$ here as well.

Considering $\langle p\rangle$, we calculate the derivative of $\psi_1(\xi)$, which is of form $K(1-\xi^2)e^{-\xi^2/2}$ for a constant K, which is even. Thus to obtain $\langle p\rangle$ we must integrate this even function multiplied by the odd function ψ_1 so the result is $\langle p\rangle = 0$.

To get the mean square values, we do the integrals:

$$\langle x^2\rangle = \frac{\hbar}{m\omega}\langle\xi^2\rangle \tag{6}$$

$$= 2\pi\left(\frac{\hbar}{m\omega}\right)^{5/2}\alpha^6\int_{-\infty}^{\infty}\xi^4 e^{-\xi^2}d\xi \tag{7}$$

$$= \frac{3}{2}\frac{\hbar}{m\omega} \tag{8}$$

And for the momentum:

$$\langle p^2\rangle = -2\pi\frac{\hbar^{5/3}}{\sqrt{m\omega}}\alpha^6\int_{-\infty}^{\infty}\xi e^{-\xi^2/2}\frac{d^2}{d\xi^2}(\xi e^{-\xi^2/2})d\xi \tag{9}$$

$$= \frac{3}{2}\hbar m\omega \tag{10}$$

For ψ_0 using the results above, the uncertainty principle here comes out to

$$\sigma_p\sigma_x = \sqrt{\langle x^2\rangle\langle p^2\rangle} = \frac{\hbar}{2} \tag{11}$$

For ψ_1, we have

$$\sigma_p\sigma_x = \sqrt{\langle x^2\rangle\langle p^2\rangle} = \frac{3\hbar}{2} \tag{12}$$

The mean kinetic and potential energies can be worked out from the above results without doing any more integration. We get$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = \hbar\omega/4$ for ψ_0 and $\langle T\rangle = \frac{\langle p^2\rangle}{2m} = 3\hbar\omega/4$ for ψ_1.

$\langle V\rangle = \frac{k\langle x^2\rangle}{2} = \hbar\omega/4$ for ψ_0 and $\langle V\rangle = \frac{k\langle x^2\rangle}{2} = 3\hbar\omega/4$ for ψ_1. Adding these together to get the total energy E gives $\hbar\omega/2$ for ψ_0 and $3\hbar\omega/2$ for ψ_1 as it should.

PINGBACKS

Pingback: Virial theorem