HARMONIC OSCILLATOR - PROBABILITY OF BEING OUTSIDE CLASSICAL REGION

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The classical harmonic oscillator has an energy of $E=\frac{1}{2}kx_0^2$ where k is the spring constant and x_0 is the maximum displacement from the equilibrium position. In terms of the frequency of oscillation, this is $E=\frac{1}{2}m\omega^2x_0^2$, so the mass oscillates between $x_0=-\sqrt{2E/m\omega^2}$ and $x_0=\sqrt{2E/m\omega^2}$. For a quantum oscillator, we can work out the probability that the particle is found outside the classical region. In the ground state, we have

$$\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar} \tag{1}$$

The probability that the particle is found between two points a and b is

$$P_{ab}=\int_a^b\psi_0^2(x)dx \tag{2}$$

so the probability that the particle is *in* the classical region is

$$P_{\text{classical}} = \int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}}\psi_0^2(x)dx \tag{3}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{2E/m\omega^2}}^{\sqrt{2E/m\omega^2}}e^{-m\omega x^2/\hbar}dx \tag{4}$$

In the ground state, $E=\hbar\omega/2$ so this is

$$P_{\text{classical}}=\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}}e^{-m\omega x^2/\hbar}dx \tag{5}$$

This is easier to deal with if we introduce a substitute variable

$$\xi\equiv\sqrt{\frac{m\omega}{\hbar}}x \tag{6}$$

Then the integral transforms to

$$P_{\text{classical}} = \frac{1}{\sqrt{\pi}} \int_{-1}^{1} e^{-\xi^2} d\xi \tag{7}$$

This integral is the *error function*, so we get (using Maple, for example):

$$\begin{aligned} P_{\text{classical}} &= \text{erf}(1) \quad (8) \\ &= 0.8427 \quad (9) \end{aligned}$$

The probability of being outside the classical region is then

$$1 - P_{\text{classical}} = 0.1573 \tag{10}$$

This is a manifestation of the quantum tunneling effect, in which a particle's wave function extends beyond the classical limit in space. Since the harmonic oscillator's potential function is a parabola, there is no cutoff point where the potential barrier becomes infinite, as in the case of the infinite square well. Thus the particle penetrates slightly into the non-classical region.