

## HARMONIC OSCILLATOR - RAISING AND LOWERING OPERATOR CALCULATIONS

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In the study of the harmonic oscillator, we can express  $x$  and  $p$  in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \quad (1)$$

$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \quad (2)$$

We now have

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \int \psi_n^*(a_+ + a_-)\psi_n dx \quad (3)$$

$$= 0 \quad (4)$$

The reason this is zero is that, as we saw when working out the normalization of the stationary states,

$$a_+\psi_n = \sqrt{n+1}\psi_{n+1} \quad (5)$$

$$a_-\psi_n = \sqrt{n}\psi_{n-1} \quad (6)$$

$$a_+a_-\psi_n = n\psi_n \quad (7)$$

$$a_--a_+\psi_n = (n+1)\psi_n \quad (8)$$

and since the wave functions are orthogonal, we get

$$\int \psi_n^*\psi_{n+1} dx = \int \psi_n^*\psi_{n-1} dx = 0 \quad (9)$$

Similarly:

$$\langle p \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \int \psi_n^*(a_+ - a_-)\psi_n dx \quad (10)$$

$$= 0 \quad (11)$$

for the same reason.

Thus we have the result that  $\langle x \rangle = \langle p \rangle = 0$  for *all* states of the harmonic oscillator. This reflects the fact that the particle oscillates about the origin  $x = 0$  with symmetric motion on both sides of the centre point.

For the mean squares:

$$\langle x^2 \rangle = \left( \frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ + a_-) (a_+ + a_-) \psi_n dx \quad (12)$$

$$= \left( \frac{\hbar}{2m\omega} \right) \int \psi_n^* (a_+ a_- + a_- a_+) \psi_n dx \quad (13)$$

$$= \left( \frac{\hbar}{2m\omega} \right) (2n + 1) \quad (14)$$

$$= \frac{\hbar}{m\omega} \left( n + \frac{1}{2} \right) \quad (15)$$

In going from the first to the second line, we've thrown out terms where we integrate two orthogonal functions. For example,

$$\int \psi_n^* a_+ a_+ \psi_n dx = \int \psi_n^* \sqrt{(n+1)(n+2)} \psi_{n+2} dx \quad (16)$$

$$= 0 \quad (17)$$

We have used the relations above and the fact that  $\psi_n$  is normalized to get the third line.

Similarly:

$$\langle p^2 \rangle = -\frac{\hbar m\omega}{2} \int \psi_n^* (-a_+ a_- - a_- a_+) \psi_n dx \quad (18)$$

$$= \hbar m\omega \left( n + \frac{1}{2} \right) \quad (19)$$

The uncertainty principle then becomes

$$\sigma_p \sigma_x = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} \quad (20)$$

$$= \hbar \left( n + \frac{1}{2} \right) \quad (21)$$

and the kinetic energy is

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{1}{2} \hbar \omega \left( n + \frac{1}{2} \right) \quad (22)$$

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which is half the total energy, as it should be.

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