

HARMONIC OSCILLATOR IN 2-D AND 3-D, AND IN POLAR AND SPHERICAL COORDINATES

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We've seen that the 3-d isotropic harmonic oscillator can be solved in rectangular coordinates using separation of variables. The Hamiltonian is

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + \frac{m\omega^2}{2} (x^2 + y^2 + z^2) \quad (1)$$

The solution to the Schrödinger equation is just the product of three one-dimensional oscillator eigenfunctions, one for each coordinate. That is

$$\psi_n(x, y, z) = \psi_{n_x}(x) \psi_{n_y}(y) \psi_{n_z}(z) \quad (2)$$

Each one-dimensional eigenfunction can be expressed in terms of Hermite polynomials as

$$\psi_{n_x}(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^{n_x} n_x!}} H_{n_x} \left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-m\omega x^2/2\hbar} \quad (3)$$

with the functions for y and z obtained by replacing x by y or z and n_x by n_y or n_z . We also saw earlier that in the 3-d oscillator, the total energy for state $\psi_n(x, y, z)$ is given in terms of the quantum numbers of the three 1-d oscillators as

$$E_n = \hbar\omega \left(n + \frac{3}{2}\right) = \hbar\omega \left(n_x + n_y + n_z + \frac{3}{2}\right) \quad (4)$$

and that the degeneracy of level n is $\frac{1}{2}(n+1)(n+2)$.

Since the Hermite polynomial H_{n_x} has parity $(-1)^{n_x}$ (that is, odd (even) polynomials are odd (even) functions), the 3-d wave function ψ_n has parity $(-1)^{n_x} (-1)^{n_y} (-1)^{n_z} = (-1)^n$.

We can write the one $n = 0$ state and three $n = 1$ states in spherical coordinates using the standard transformation

$$x = r \sin \theta \cos \phi \quad (5)$$

$$y = r \sin \theta \sin \phi \quad (6)$$

$$z = r \cos \theta \quad (7)$$

Using the notation $\psi_n = \psi_{n_x n_y n_z} = \psi_{n_x} \psi_{n_y} \psi_{n_z}$, we have, using $H_0(y) = 1$ and $H_1(y) = 2y$:

$$\psi_{000} = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} \quad (8)$$

$$\psi_{100} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin \theta \cos \phi \quad (9)$$

$$\psi_{010} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \sin \theta \sin \phi \quad (10)$$

$$\psi_{001} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-m\omega r^2/2\hbar} r \cos \theta \quad (11)$$

We can check that these are the correct spherical versions of the eigenfunctions by using the Schrödinger equation in spherical coordinates, which is

$$H\psi = \left[-\frac{\hbar^2 \nabla^2}{2m} + \frac{m\omega^2}{2} r^2 \right] \psi = E\psi \quad (12)$$

The spherical Laplacian operator is

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \quad (13)$$

You can grind through the derivatives by hand if you like, but I just used Maple to do it, giving the results

$$H\psi_{000} = \frac{3}{2} \hbar \omega \psi_{000} \quad (14)$$

$$H\psi_{100} = \frac{5}{2} \hbar \omega \psi_{100} \quad (15)$$

$$H\psi_{010} = \frac{5}{2} \hbar \omega \psi_{010} \quad (16)$$

$$H\psi_{001} = \frac{5}{2} \hbar \omega \psi_{001} \quad (17)$$

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In two dimensions, the analysis is pretty much the same. In the more general case where the masses are equal, but $\omega_x \neq \omega_y$, the Hamiltonian is

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m}{2} (\omega_x^2 x^2 + \omega_y^2 y^2) \quad (18)$$

A solution by separation of variables still works, with the result

$$\psi_n(x, y) = \psi_{n_x}(x) \psi_{n_y}(y) \quad (19)$$

The total energy is

$$E_n = E_{n_x} + E_{n_y} = \hbar\omega \left(n_x + \frac{1}{2} + n_y + \frac{1}{2} \right) = \hbar\omega (n + 1) \quad (20)$$

For a given energy level $n = n_x + n_y$, there are $n + 1$ ways of forming n out of a sum of 2 non-negative integers, so the degeneracy of level n is $n + 1$.

The one $n = 0$ state and two $n = 1$ states are

$$\psi_{00} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar} \quad (21)$$

$$\psi_{10} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar} x \quad (22)$$

$$\psi_{01} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega(x^2+y^2)/2\hbar} y \quad (23)$$

To translate to polar coordinates, we use the transformations

$$x = \rho \cos \phi \quad (24)$$

$$y = \rho \sin \phi \quad (25)$$

so we have

$$\psi_{00} = \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar} \quad (26)$$

$$\psi_{10} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar} \rho \cos \phi \quad (27)$$

$$\psi_{01} = \sqrt{\frac{2m\omega}{\hbar}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/2} e^{-m\omega\rho^2/2\hbar} \rho \sin \phi \quad (28)$$

Again, we can check this by plugging these polar formulas into the polar Schrödinger equation, where the 2-d Laplacian is

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$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} \quad (29)$$

The results are

$$H\psi_{00} = \hbar\omega\psi_{00} \quad (30)$$

$$H\psi_{10} = 2\hbar\omega\psi_{10} \quad (31)$$

$$H\psi_{01} = 2\hbar\omega\psi_{01} \quad (32)$$

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