LAGRANGIAN FOR CLASSICAL ELECTROMAGNETISM

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The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=0 \tag{1}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. For systems where the potential energy $V(q_i)$ is independent of the velocities $\dot{q}_i$, the Lagrangian can be written as

$$L=T-V \tag{2}$$

where T is the kinetic energy. However, there is one important area in classical physics where the potential *does* depend on velocity, and that is electromagnetism.

The relation between the electric scalar potential ϕ, the magnetic vector potential **A** and the electric and magnetic fields **E** and **B** is given by Maxwell's equations in terms of potentials:

$$\mathbf{E}=-\nabla\phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \tag{3}$$

$$\mathbf{B}=\nabla\times\mathbf{A} \tag{4}$$

[These are the forms used by Shankar, which are in Gaussian units. Many of my posts on electromagnetism are taken from Griffiths's book, which uses the MKS system of units, so various constants will be different in the two systems.]

The force on a charge q due to electric and magnetic fields **E** and **B** is given by

$$\mathbf{F}=q\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right) \tag{5}$$

Shankar merely states that the correct force can be derived from 1 if we use the Lagrangian

$$L=\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}-q\phi+\frac{q}{c}\mathbf{v}\cdot\mathbf{A} \tag{6}$$

It appears from a bit of googling that this Lagrangian is obtained more or less by trial and error, rather than by some rigorous derivation, so it seems we just need to accept it "because it works". The velocity **v** in rectangular coordinates is

$$\mathbf{v} = [\dot{x}_1, \dot{x}_2, \dot{x}_3] \tag{7}$$

$$\mathbf{v}\cdot\mathbf{v} = \sum_{i=1}^{3} \dot{x}_i^2 \tag{8}$$

$$\mathbf{v}\cdot\mathbf{A} = \sum_{i=1}^{3} \dot{x}_i A_i \tag{9}$$

Both ϕ and **A** are functions of position, so depend on x_i. Thus from 1, we have

$$\frac{d}{dt}\left(m\dot{x}_i + \frac{q}{c}A_i\right) = -q\frac{\partial\phi}{\partial x_i} + \frac{q}{c}\frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial x_i} \tag{10}$$

The three equations represented here can be combined into a single vector equation by noticing that $\frac{\partial}{\partial x_i}$ are the components of the gradient.

$$\frac{d}{dt}\left(m\mathbf{v} + \frac{q}{c}\mathbf{A}\right) = -q\nabla\phi + \frac{q}{c}\nabla(\mathbf{v}\cdot\mathbf{A}) \tag{11}$$

The LHS contains the total time derivative $\frac{d\mathbf{A}}{dt}$ which is composed of two contributions. First, **A** itself can be time varying, in the sense that if we stayed at the same location, the value of **A** at that location can vary in time. The second contribution comes from the motion of the charge so that, even if **A** is constant in time, the charge will perceive a change in **A** as it moves because **A** can vary over space. That is, the total derivative of the first component A_1 is

$$\frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + \sum_{i=1}^{3}\frac{\partial A_1}{\partial x_i}\frac{dx_i}{dt} \tag{12}$$

$$= \frac{\partial A_1}{\partial t} + (\mathbf{v}\cdot\nabla)A_1 \tag{13}$$

The derivative of **A** can thus be written as

$$\frac{d\mathbf{A}}{dt} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{14}$$

Plugging this into 11 and rearranging, we get

$$\frac{d}{dt}(m\mathbf{v}) = -q\nabla\phi - \frac{q}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{q}{c}\left[\nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}\right] \tag{15}$$

$$\mathbf{F} = -q\nabla\phi - \frac{q}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{q}{c}(\mathbf{v}\times(\nabla\times\mathbf{A})) \tag{16}$$

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) \tag{17}$$

In the second line, we used a standard vector identity:

$$\mathbf{v}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A} \tag{18}$$

Thus the Lagrangian 6 does indeed give the correct force law. The Lagrangian is not of the form $T-V$ because the term $q\phi - \frac{q}{c}\mathbf{v}\cdot\mathbf{A}$ isn't a potential energy. In electrostatics, $q\phi$ is indeed potential energy, but because the magnetic force always acts perpendicular to the velocity, it does no work, so we can't interpret $-\frac{q}{c}\mathbf{v}\cdot\mathbf{A}$ as some form of 'magnetic potential energy'. The work done when moving a charge through an electromagnetic field in general depends on the path taken, so is not conservative, and we can't write the force as the gradient of some potential.