MOMENTUM EIGENFUNCTIONS AND EIGENVALUES

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The momentum operator in one dimension is

$$\hat{p}=-i\hbar d/dx \tag{1}$$

Since momentum is an observable quantity, its operator should be hermitian. To show this, we must show that

$$\left\langle \phi\left|\hat{p}\psi\right.\right\rangle =\left\langle \hat{p}\phi\left|\psi\right.\right\rangle \tag{2}$$

for some functions that tend to zero at infinity. Writing this in terms of integrals, we have

$$\begin{aligned}
\left\langle \phi\left|\hat{p}\psi\right.\right\rangle &= -i\hbar\int\phi^{*}\frac{d\psi}{dx}dx &(3)\\
&= -i\hbar\left[\phi\psi\right]+i\hbar\int\frac{d\phi^{*}}{dx}\psi dx &(4)\\
&= i\hbar\int\frac{d\phi^{*}}{dx}\psi dx &(5)\\
&= \left\langle \hat{p}\phi\left|\psi\right.\right\rangle &(6)
\end{aligned}$$

where in the second line, we used the condition that ϕ and ψ are zero at infinity, which are the limits of the integration.

However, if we try to find the eigenvalues and eigenfunctions of $\hat{p}$, we run into a bit of a problem. We try to solve, for some eigenvalue p:

$$\begin{aligned}
\hat{p}f &= pf &(7)\\
-i\hbar\frac{d}{dx}f &= pf &(8)
\end{aligned}$$

This has the formal solution

$$f_{p}(x)=Ae^{ipx/\hbar} \tag{9}$$

for some constant A. Ordinarily, at this stage, we would impose some boundary condition on the solution to obtain acceptable values of p. The problem is that we'd like to define this function over all x and, if we try to do this, the function is not normalizable for any value of p. At first glance, we might think that if we chose p to be purely imaginary as in $p = \alpha i$, it might work since we get

$$f(x) = Ae^{-\alpha x/\hbar} \tag{10}$$

but this tends to infinity at large negative x so that doesn't work. In fact if p has a non-zero imaginary part, $f(x)$ goes to infinity at one end of its domain. So we're restricted to looking at real values of p.

In that case, $f(x)$ is periodic and thus is still not normalizable. Thus there are no eigenfunctions of the momentum operator that lie in Hilbert space (which, remember, is the vector space of square-integrable functions).

What happens if do the normalization integral anyway? That is, we try

$$\int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x)\, dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p_2-p_1)x/\hbar} dx \tag{11}$$

By using the variable transformation $\xi \equiv x/\hbar$, we get

$$\int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x)\, dx = |A|^2 \hbar \int_{-\infty}^{\infty} e^{i(p_2-p_1)\xi} d\xi \tag{12}$$

It's at this point that we invoke the Fourier transform involving the Dirac delta function that we obtained a while back. Using this, we can write the integral as a delta function, and we get

$$\int_{-\infty}^{\infty} f_{p_1}^*(x) f_{p_2}(x)\, dx = 2\pi |A|^2 \hbar \delta(p_2 - p_1) \tag{13}$$

This is sort of like a normalization condition, in that the integral is zero when $p_1 \neq p_2$, and non-zero (infinite, in fact) if $p_1 = p_2$. In fact, if we take the constant A to be

$$A = \frac{1}{\sqrt{2\pi\hbar}} \tag{14}$$

and use the bra-ket notation for the integral, we can write

$$\langle f_{p_1} | f_{p_2} \rangle = 2\pi \frac{1}{2\pi\hbar} \hbar \delta(p_2 - p_1) \tag{15}$$

$$= \delta(p_2 - p_1) \tag{16}$$

We can also express an arbitrary function $g(x)$ as a Fourier transform over p by writing

$$g(x) = \int_{-\infty}^{\infty} c(p)\, f_p(x)\, dp \tag{17}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p)\, e^{ipx/\hbar} dp \tag{18}$$

$$g(\hbar\xi) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p)\, e^{ip\xi} dp \tag{19}$$

From Plancherel's theorem, we can invert this relation to get $c(p)$:

$$c(p) = \sqrt{\frac{\hbar}{2\pi}} \int_{-\infty}^{\infty} g(\hbar\xi)\, e^{-ip\xi} d\xi \tag{20}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} g(x)\, e^{-ipx/\hbar} dx \tag{21}$$

$$= \langle f_p | g \rangle \tag{22}$$

With these definitions, the eigenfunctions of the momentum operator are therefore

$$\frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \tag{23}$$

In general, hermitian operators with continuous eigenvalues don't have normalizable eigenfunctions and have to be analyzed in this way. In particular, the hamiltonian (energy) of a system can have an entirely discrete spectrum (infinite square well or harmonic oscillator), a totally continuous spectrum (free particle, delta function barrier or finite square barrier) or a mixture of the two (delta function well or finite square well).

REFERENCES

(1) Berman, Paul R. (2018), *Introductory Quantum Mechanics*, Springer, Chapter 5.
(2) Griffiths, David J. (2005), *Introduction to Quantum Mechanics*, 2nd Edition; Pearson Education, Chapters 1 & 3.
(3) Landau, L.D. & Lifshitz, E.M. (1977), *Quantum Mechanics (Non-relativistic Theory)*, 3rd Edition, Butterworth-Heineman, §15.

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