MOMENTUM SPACE

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Although we usually work in *position space* when solving the Schrödinger equation since we want the wave function as a position of spatial coordinates and time, there is an alternative way of looking at quantum mechanical quantities and operators known as *momentum space*.

The momentum space wave function is the Fourier transform of the regular position space wave function:

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ipx/\hbar}dx \tag{1}$$

with the original position space function being the inverse transform of the momentum space version:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\Phi(p,t)e^{ipx/\hbar}dp \tag{2}$$

As an illustration, we can find the forms of the position and momentum operators in momentum space.

The position operator $\hat{x}$, in position space, merely multiplies the wave function by the value x of the position.

$$\hat{x}\psi(x,t) = x\psi(x,t) \tag{3}$$

To see what form $\hat{x}$ takes in momentum space, we can calculate the mean position $\langle x\rangle$.The mean position turns out to be

$$\langle x\rangle = \int\Phi^*\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\Phi dp \tag{4}$$

We can show this as follows. Starting with the definition of Φ in 1, we get

$$-\frac{\hbar}{i}\frac{\partial}{\partial p}\Phi = \frac{1}{\sqrt{2\pi\hbar}}\int xe^{-ipx/\hbar}\Psi dx \tag{5}$$

Substituting this together with the expression for Φ^* from 1 into the RHS of 4 we get:

$$\int \mathbf{\Phi}^* \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\mathbf{\Phi} dp = \frac{1}{2\pi\hbar}\int \left(\int e^{ipx'/\hbar}\mathbf{\Psi}^*(x')dx'\right)\left(\int x e^{-ipx/\hbar}\mathbf{\Psi}(x)dx\right)dp \tag{6}$$

$$= \frac{1}{2\pi\hbar}\int \left(\int\int e^{ip(x'-x)/\hbar}\mathbf{\Psi}^*(x')x\mathbf{\Psi}(x)dx'dx\right)dp \tag{7}$$

We can now do the integral over p first, and use the Fourier transform form of the Dirac delta function we obtained earlier, which expressed a delta function as an integral over a complex exponential:

$$\frac{1}{2\pi\hbar}\int e^{ip(x'-x)/\hbar}dp = \delta(x'-x) \tag{8}$$

Thus:

$$\int \mathbf{\Phi}^* \left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\mathbf{\Phi} dp = \int\int \delta(x'-x)\mathbf{\Psi}^*(x')x\mathbf{\Psi}(x)dx'dx \tag{9}$$

$$= \int \mathbf{\Psi}^*(x)x\mathbf{\Psi}(x)dx \tag{10}$$

$$= \langle x\rangle \tag{11}$$

Thus we have the two forms of the position operator:

$$\underbrace{\hat{x}}_{\text{position space}} \rightarrow \underbrace{i\hbar\frac{\partial}{\partial p}}_{\text{momentum space}} \tag{12}$$

Because the form of the position operator in momentum space is identical (apart from a sign) to the form of the momentum operator in position space, the two versions of the eigenfunctions of the position operator are

$$\begin{aligned} \delta\left(x-x'\right) & \quad \text{position space} \\ \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar} & \quad \text{momentum space} \end{aligned} \tag{13}$$

In momentum space, the mean value of the momentum is simply

$$\langle p\rangle = \langle \mathbf{\Phi}|\, p\mathbf{\Phi}\rangle \tag{14}$$

$$= \int_{-\infty}^{\infty}\mathbf{\Phi}^* p\mathbf{\Phi} dp \tag{15}$$

Thus the two forms of the momentum operator are

$$\underbrace{-i\hbar\frac{\partial}{\partial x}}_{\text{position space}} \rightarrow \underbrace{\hat{p}}_{\text{momentum space}} \tag{16}$$

The eigenfunctions of the momentum operator are

$$\begin{aligned} \delta\left(p-p'\right) & \text{ momentum space} \\ \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} & \text{ position space} \end{aligned} \tag{17}$$

References

(1) Berman, Paul R. (2018), *Introductory Quantum Mechanics*, Springer, Chapter 5.
(2) Griffiths, David J. (2005), *Introduction to Quantum Mechanics*, 2nd Edition; Pearson Education, Chapter 3.