PARITY TRANSFORMATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 19 July 2021.

A parity transformation reflects all the coordinate axes through the origin, so that, in one dimension $x\to-x$ and in three dimensions the position vector $\mathbf{r}\to-\mathbf{r}$. In one dimension, a parity transformation is the same as reflection in a point-sized mirror placed at the origin. It might seem that in three dimensions, parity is more than just a reflection in a plane mirror, but in fact it can be shown that it is equivalent to such a reflection followed by a rotation. To see this, suppose we place a mirror in the xy plane, so that the z axis gets reflected into $-z$. This converts a right-handed rectangular coordinate system (where the direction of the z axis is determined by the direction of your thumb on your right hand when you curl your fingers through the right angle between the positive x and y axes) into a left-handed coordinate system (the direction of the new $+z$ axis is found by doing the finger-curling maneuver with your left hand). However, merely reflecting the z axis in the xy plane leaves the x and y axes unchanged. Now if we rotate the xy plane by an angle π (or 180°) about the z axis, then the $+x$ axis gets rotated into the $-x$ axis, and the $+y$ axis gets rotated into the $-y$ axis. In this sense, the 3-d parity transformation is equivalent to a reflection (since pretty well every physical phenomenon is invariant under a rotation).

To apply parity to quantum state vectors, we define a parity operator Π to have the following action on the X basis:

$$\Pi\left|x\right\rangle =\left|-x\right\rangle \tag{1}$$

From this definition we can see the effect on an arbitrary state $\left|\psi\right\rangle$ by inserting a complete set of X states:

$$\Pi\left|\psi\right\rangle = \Pi\int_{-\infty}^{\infty}\left|x\right\rangle\left\langle x\left|\psi\right.\right\rangle dx \tag{2}$$

$$= \int_{-\infty}^{\infty}\left|-x\right\rangle\left\langle x\left|\psi\right.\right\rangle dx \tag{3}$$

$$= \int_{\infty}^{-\infty}\left|x'\right\rangle\left\langle -x'\left|\psi\right.\right\rangle\left(-dx'\right) \tag{4}$$

$$= \int_{-\infty}^{\infty}\left|x'\right\rangle\left\langle -x'\left|\psi\right.\right\rangle dx' \tag{5}$$

In the third line we made the substitution $x' = -x$, so that $dx = -dx'$ and the limits of integration get swapped. As a result of this, the effect of parity in the X basis representation $\left\langle x\left|\psi\right.\right\rangle = \psi(x)$ of a state vector $\left|\psi\right\rangle$ is

$$\left\langle x\left|\Pi\right|\psi\right\rangle = \int_{-\infty}^{\infty}\left\langle x\left|x'\right.\right\rangle\left\langle -x'\left|\psi\right.\right\rangle dx' \tag{6}$$

$$= \int_{-\infty}^{\infty}\delta\left(x-x'\right)\left\langle -x'\left|\psi\right.\right\rangle dx' \tag{7}$$

$$= \psi(-x) \tag{8}$$

Parity therefore simply converts $x \to -x$ wherever it occurs in the function $\psi(x)$.

One special case of this is the momentum eigenstate $\left|p\right\rangle$ which has the form in the X basis of

$$\left\langle x\left|p\right.\right\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar} \tag{9}$$

The parity transformation gives

$$\left\langle x\left|\Pi\right|p\right\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-ipx/\hbar} \tag{10}$$

Another way of looking at this is that parity changes p to $-p$ and leaves the x alone, so that

$$\Pi\left|p\right\rangle = \left|-p\right\rangle \tag{11}$$

[You might think that if parity transforms $x \to -x$ and $p \to -p$ then the effect on $e^{ipx/\hbar}$ should be to switch the signs of both x and p and thus leave the state unchanged. However, this isn't correct, as we can express a state vector in either the X basis (in which $x \to -x$) *or* in the P basis (in which $p \to -p$) but *not both at the same time*.]

A few properties of Π can be derived fairly easily. First, since applying Π twice in succession to the same state swaps $x \to -x$ and back again, it leaves that state unchanged. Since this is true for all states, we must have

$$\Pi^2 = I \tag{12}$$

from which we see that Π is its own inverse, so

$$\Pi^{-1} = \Pi \tag{13}$$

We can also see that Π is Hermitian by considering

$$\left\langle \psi \left| \Pi^\dagger \Pi \right| \psi \right\rangle = \langle \Pi\psi | \Pi\psi \rangle = \int_{-\infty}^{\infty} \psi^* (-x) \psi (-x) \, dx \tag{14}$$

$$= \int_{-\infty}^{\infty} \psi^* \left(x'\right) \psi \left(x'\right) dx' \tag{15}$$

$$= \langle \psi | \psi \rangle \tag{16}$$

In the second line we used the same trick as in the derivation of 5 to substitute $x' = -x$. Thus we see that

$$\Pi^\dagger \Pi = I \tag{17}$$

$$\Pi^\dagger = \Pi^{-1} = \Pi \tag{18}$$

The condition $\Pi^\dagger = \Pi$ shows that Π is Hermitian, and the condition $\Pi^\dagger = \Pi^{-1}$ shows that Π is unitary.

Finally, any operator whose square is the identity operator has eigenvalues ± 1, as we can see as follows. Suppose $|\psi\rangle$ is an eigenvector of Π with eigenvalue α. Then

$$\Pi |\psi\rangle = \alpha |\psi\rangle \tag{19}$$

$$\Pi^2 |\psi\rangle = \alpha \Pi |\psi\rangle \tag{20}$$

$$= \alpha^2 |\psi\rangle \tag{21}$$

$$= I |\psi\rangle \tag{22}$$

$$= |\psi\rangle \tag{23}$$

Therefore $\alpha^2 = 1$, so $\alpha = \pm 1$.

We can also define Π by examining its effect on operators, rather than states. Consider

$$\left\langle \Pi x' \left| X \right| \Pi x \right\rangle = \left\langle -x' \left| X \right| -x \right\rangle \tag{24}$$

$$= -x\delta\left(x' - x\right) \tag{25}$$

However, this is equivalent to

$$\left\langle \Pi x' \left| X \right| \Pi x \right\rangle = \left\langle x' \left| \Pi^{\dagger} X \Pi \right| x \right\rangle = -x\delta\left(x' - x\right) \tag{26}$$

Thus we can write

$$\Pi^{\dagger} X \Pi = -X \tag{27}$$

and similarly for the momentum

$$\Pi^{\dagger} P \Pi = -P \tag{28}$$

Eigenstates of parity are said to be even if the eigenvalue is $+1$ and odd if the eigenvalue is -1. Mathematically, the X basis representation of such eigenstates are even or odd functions of x, respectively.

The Hamiltonian is parity invariant if a parity transformation leaves it unchanged, so that

$$\Pi^{\dagger} H(X,P)\,\Pi = H(-X,-P) = H(X,P) \tag{29}$$

Since $\Pi^{\dagger} = \Pi$, this condition is equivalent to

$$[\Pi, H] = 0 \tag{30}$$

Using the same argument as with conservation of momentum, if this commutator is valid at all times (if H is time-independent this is automatic; if H is time-dependent, then we must impose the commutator at all times), then Π must also commute with the propagator $U(t)$, since U depends only on H. In this case, if we start with a system in a definite parity state (even or odd), then the parity of the state doesn't change with time. This follows because if $[\Pi, U(t)] = 0$ then if $\Pi\left|\psi(0)\right\rangle = \alpha\left|\psi(0)\right\rangle$ (where $\alpha = \pm 1$), then we can let the state evolve in time by applying the propagator to it, so that we have

$$\left|\psi(t)\right\rangle = U(t)\left|\psi(0)\right\rangle \tag{31}$$

Applying the parity operator to this and using the commutator, we have

$$\Pi\left|\psi(t)\right\rangle = \Pi U(t)\left|\psi(0)\right\rangle = U(t)\,\Pi\left|\psi(0)\right\rangle = \alpha U(t)\left|\psi(0)\right\rangle = \alpha\left|\psi(t)\right\rangle \tag{32}$$

Thus the parity of the evolved state is the same as the parity of the initial state.

Parity is not always conserved in physics. A notable parity-violating reaction is a decay involving the weak nuclear force. Shankar describes one such case with the decay of an isotope of cobalt: $^{60}\mathrm{Co} \rightarrow^{60}\mathrm{Ni} + e^{-} + \bar{\nu}$. Another example is in Shankar's exercise 11.4.3.

Suppose that in one particular reaction which emits an electron, the electron's spin is observed to be always parallel to its momentum. For the purposes of this argument, we can regard an electron's spin as being caused by some physical rotation of the electron. Suppose in one such reaction, the electron's spin is in the $+z$ direction (using the right-hand rule for calculating the direction of angular momentum, so that viewed from above, the electron is rotating counterclockwise) and therefore its momentum is also in the $+z$ direction. Now reflect this reaction in a mirror lying in the yz plane. This reflection will invert the direction of rotation (think of viewing a spinning top in a mirror) so that the spin direction will now point in the $-z$ direction, but since the momentum vector is parallel to the plane of the mirror, it will *not* be inverted. Thus the spin and momentum are now antiparallel after a parity transformation, showing that parity in this case is not conserved.

Finally, Shankar includes a curious problem (11.4.2) which, as far as I can tell, doesn't have anything to do with parity, but I'll include it here for completeness. Suppose we have a particle that moves in a potential

$$V(x) = V_0 \sin\left(\frac{2\pi x}{a}\right) \tag{33}$$

This potential is periodic with a period of a, so if we translate the system according to $x \rightarrow x + ma$ for some integer m, the potential is unchanged. The problem is to show that momentum is not conserved in this case. The conservation of momentum argument, valid for infinitesimal translations, relied on Ehrenfest's theorem, which states that

$$\left\langle \dot{P} \right\rangle = -\frac{i}{\hbar} \langle [P, H] \rangle \tag{34}$$

If the momentum commutes with the Hamiltonian, then, on average, the momentum is conserved. Now in this case we can calculate the commutator $[P, V]$ using the result

$$[X^n, P] = i\hbar n X^{n-1} \tag{35}$$

We can write the potential as a series:

$$V(X) = V_0 \left[\frac{2\pi X}{a} - \frac{1}{3!} \left(\frac{2\pi X}{a} \right)^3 + \ldots \right] \tag{36}$$

The commutator is therefore

$$[V,P]=\frac{2\pi i\hbar V_{0}}{a}\left[1-\frac{1}{2!}\left(\frac{2\pi X}{a}\right)^{2}+\ldots\right]=\frac{2\pi i\hbar V_{0}}{a}\cos\left(\frac{2\pi X}{a}\right) \tag{37}$$

Therefore, Ehrenfest's theorem gives us (since H presumably is of the form $H=T+V$ with the kinetic energy depending only on P, so it commutes with P):

$$\left\langle \dot{P}\right\rangle =-\frac{2\pi V_{0}}{a}\left\langle \cos\left(\frac{2\pi X}{a}\right)\right\rangle \tag{38}$$

Since the cosine is periodic, we can't actually calculate a unique value for its average, although if we do the average over an exact number of periods, the average is still zero. I have a feeling that I'm missing something obvious here, so any suggestions are welcome.

PINGBACKS

Pingback: Time reversal, antiunitary operators and Wigner's theorem