PHASES IN THE ADIABATIC APPROXIMATION

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The adiabatic theorem (see Griffiths, section 10.1 for a proof) says that if a system starts out in the nth state of a time-dependent hamiltonian, and the hamiltonian changes slowly compared to the internal period of the time-independent wave function (that is, the time scale over which the hamiltonian changes is much longer than $\hbar/E_n$), then after a time t the system will end up in state

$$\boldsymbol{\Psi}_n(t) = e^{i\theta_n(t)}e^{i\gamma_n(t)}\psi_n(t) \tag{1}$$

where

$$\theta_n(t) \equiv -\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \tag{2}$$

$$\gamma_n(t) \equiv i\int_0^t \left\langle \psi_n(t') \left| \frac{\partial}{\partial t'}\psi_n(t') \right.\right\rangle dt' \tag{3}$$

θ is called the *dynamic phase* and γ is called the *geometric phase*.

The wave functions $\psi_n(t)$ are the solutions of the eigenvalue equation at a particular time t:

$$H(t)\psi_n(t) = E_n(t)\psi_n(t) \tag{4}$$

That is, they aren't a full solution of the time dependent Schrödinger equation; rather they are the solutions of the time-independent Schrödinger equation with whatever parameters are now time-dependent in the hamiltonian replaced by their time-dependent forms.

For example, with an infinite square well whose right wall moves so that its position w is a function of time $w(t)$, we have

$$\psi_n(t) = \sqrt{\frac{2}{w(t)}}\sin\frac{n\pi}{w(t)}x \tag{5}$$

$$E_n(t) = \frac{(n\pi\hbar)^2}{2mw^2(t)} \tag{6}$$

In this case, ψ_n depends on only one time-dependent parameter, so we can use the chain rule to write

$$\gamma_n(t) = i\int_0^t \left\langle \psi_n \middle| \frac{\partial}{\partial w}\psi_n \right\rangle \frac{dw}{dt'}dt' \tag{7}$$

$$= i\int_{w_1}^{w_2} \left\langle \psi_n \middle| \frac{\partial}{\partial w}\psi_n \right\rangle dw \tag{8}$$

where the wall moves from w_1 to w_2 between times 0 and t. We get

$$\frac{\partial}{\partial w}\psi_n = -\frac{\sqrt{2}}{2w^{5/2}}\left[w\sin\frac{n\pi}{w}x + 2n\pi x\cos\frac{n\pi}{w}x\right] \tag{9}$$

$$\left\langle \psi_n \middle| \frac{\partial}{\partial w}\psi_n \right\rangle = -\frac{1}{w^3}\int_0^w \sin\frac{n\pi}{w}x\left[w\sin\frac{n\pi}{w}x + 2n\pi x\cos\frac{n\pi}{w}x\right]dx \tag{10}$$

$$= \frac{\sin^2 n\pi}{w} \tag{11}$$

$$= 0 \tag{12}$$

In this case, there is no change in phase due to the geometric phase. In fact, we can see this is generally true for real wave functions ψ_n since

$$\langle \psi_n | \psi_n \rangle = 1 \tag{13}$$

$$\frac{d}{dt}\langle \psi_n | \psi_n \rangle = 0 \tag{14}$$

$$= \left\langle \dot{\psi}_n \middle| \psi_n \right\rangle + \left\langle \psi_n \middle| \dot{\psi}_n \right\rangle \tag{15}$$

$$= 2\Re\left(\left\langle \psi_n \middle| \dot{\psi}_n \right\rangle\right) \tag{16}$$

That is, $\left\langle \psi_n(t') \middle| \frac{\partial}{\partial t'}\psi_n(t') \right\rangle$ must be purely imaginary, so if ψ_n is real, the bracket must be zero. This also means that γ is always real.Thus γ is zero as the wall moves from w_1 to w_2 and also as it moves back from w_2 to w_1.

The dynamic phase for the same journey is

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \tag{17}$$

$$= -\frac{\hbar(n\pi)^2}{2m}\int_0^t \frac{1}{w^2(t')}dt' \tag{18}$$

If the speed of the wall is constant so that $w = w_1 + vt$ we have

$$\theta_n(t) = -\frac{\hbar(n\pi)^2}{2m}\int_0^{(w_2-w_1)/v}\frac{dt'}{(w_1+vt')^2} \tag{19}$$

$$= \frac{\hbar(n\pi)^2}{2mv}\frac{w_1-w_2}{w_1 w_2} \tag{20}$$

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