POSTULATES OF QUANTUM MECHANICS - STATES AND MEASUREMENTS

Different books give different sets of postulates on which non-relativistic quantum mechanics is based. Most sources that I’ve seen, however, do tend to agree on the essentials, so we’ll run through those here.

In classical mechanics, the path of a particle is, in the Hamiltonian formalism, described by specifying its position $x(t)$ and momentum $p(t)$ as functions of time. Both the position and momentum are specified precisely at all times. In quantum mechanics, the state of a particle is specified by a vector (ket) $|\psi(t)\rangle$ in a Hilbert space. This vector can be expressed either in position space as a wave function $\psi(\mathbf{r},t)$ or in momentum space as the Fourier transform of $\psi$:

$$
\phi(\mathbf{p},t) = \int d^3r \, \psi(\mathbf{r},t) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} \tag{1}
$$

The probability of finding the particle in the infinitesimal spatial volume $d^3r$ centred at location $\mathbf{r}$ is $|\psi(\mathbf{r},t)|^2 \, d^3r$, and the probability of the particle’s momentum being in the infinitesimal volume $d^3p$ centred at momentum $\mathbf{p}$ is $|\phi(\mathbf{p},t)|^2 \, d^3p$.

The time dependence of $\psi(\mathbf{r},t)$ is assumed to be given by the time-dependent Schrödinger equation

$$
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\mathbf{r}) \psi \tag{2}
$$

In classical mechanics, any dynamical variable $\omega$ is a function of the two phase-space coordinates $x$ and $p$: $\omega = \omega(x,p)$. In quantum mechanics, the spatial coordinate $x$ is replaced by a Hermitian operator $X$ and the momentum $p$ is replaced by the differential operator $P = -i\hbar \nabla$ which we discussed earlier. The classical dynamical variable $\omega(x,p)$ becomes a Hermitian operator $\Omega(X,P)$, where $x$ and $p$ in $\omega(x,p)$ are replaced by their corresponding operators $X$ and $P$.

In classical mechanics, it is assumed that (in principle) any dynamical variable $\omega$ may be measured with arbitrary precision without changing the state of the particle. In quantum mechanics, every physically measurable
quantity is assumed to have a Hermitian operator associated with it. If we wish to measure the value of a variable represented by the Hermitian operator $\Omega$, we must determine the eigenvalues $\omega_i$ and corresponding eigenvectors $|\omega_i\rangle$ of $\Omega$, then express the state $|\psi\rangle$ as a linear combination of the $|\omega_i\rangle$. Then the best we can do is to state that the particular eigenvalue $\omega_i$ will be measured with probability $|\langle \omega_i | \psi \rangle|^2$. After the measurement, the state $|\psi\rangle$ 'collapses' to become the state $|\omega_i\rangle$. The only possible outcomes of a measurement of $\Omega$ are its eigenvalues; no intermediate values are possible. The collapse of the wave function is an ideal case. In practice, if a measurement of a physical quantity (associated with the Hermitian operator $\Omega$, say) produces a value $\omega_i$, the wave function (state vector) of the system is changed, but it may not actually become an eigenvector of $\Omega$ due to side effects of the measurement. For example, if we have a particle in a state $|x\rangle$ which is an eigenstate of the position operator $X$, then the idea of collapse of the wave function might lead us to expect that a measurement of position would yield precisely $x$ and leave the system unchanged, since it was already in an eigenstate of position. However, in order to measure the position of a particle, we need interact with the particle, perhaps by bouncing a photon off it. A photon can determine the position of something only to within a location roughly equal to its wavelength, so to get a precise position measurement, we would need a photon of very short wavelength, which corresponds to a very high frequency, and thus to a very large momentum. Bouncing such a photon off a particle is bound to have some effect on the particle’s state.

To illustrate these postulates, suppose we have the following three operators on a complex 3-d Hilbert space:

$$L_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$  \quad (3)

$$L_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$  \quad (4)

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$  \quad (5)

Since $L_z$ is diagonal, its eigenvalues can be read off from the diagonal elements as $0, \pm 1$, so these are the possible values of $L_z$ that could be obtained in a measurement. Also because $L_z$ is diagonal, its eigenvectors are
Suppose we start with the state $|L_z = +1\rangle$ in which $L_z = +1$, and we want to measure $L_x$ in this state. To find the expectation values $\langle L_x \rangle$ and $\langle L_x^2 \rangle$ in this state, we calculate

$$\langle L_x \rangle = \langle L_z = +1 | L_x | L_z = +1 \rangle$$

$$= [ 1 \ 0 \ 0 ] \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} [ 1 \ 0 \ 0 ] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= 0$$

To get $\langle L_x^2 \rangle$ we first find the operator

$$L_x^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now we have

$$\langle L_x^2 \rangle = [ 1 \ 0 \ 0 ] \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} [ 1 \ 0 \ 0 ] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}$$
The uncertainty, or variance, is

\[ \Delta L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} = \frac{1}{\sqrt{2}} \]  \hspace{1cm} (17)

To find the possible values of \( L_x \) and their probabilities, we need to find the eigenvalues and eigenvectors of \( L_x \), which we can do in the \( L_z \) basis, since this basis is given by the three vectors in 6. The eigenvalues are found in the usual way from the determinant:

\[
\begin{vmatrix}
-\lambda & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\lambda
\end{vmatrix} = -\lambda \left( \lambda^2 - \frac{1}{2} \right) - \frac{1}{\sqrt{2}} \left( \frac{-\lambda}{\sqrt{2}} \right) \]

\[ = -\lambda^3 + \lambda = 0 \]  \hspace{1cm} (18)

\[ \lambda = 0, \pm 1 \]  \hspace{1cm} (19)

The eigenvectors can be found in the usual way, by solving

\[(L_x - \lambda I) |L_x = \lambda \rangle = 0 \]  \hspace{1cm} (21)

where the ket takes on the three possible values of \( \lambda \) successively. We let

\[ |L_x = \lambda \rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]  \hspace{1cm} (22)

For \( \lambda = +1 \) we have

\[ \begin{align*}
-a + \frac{b}{\sqrt{2}} &= 0 \\
\frac{1}{\sqrt{2}} (a - \sqrt{2}b + c) &= 0 \\
b - c &= 0
\end{align*} \]  \hspace{1cm} (23-25)

Only two of these three equations are independent, so we can set \( a = 1 \) and solve for \( b \) and \( c \) to get

\[ \begin{align*}
a &= 1 \\
b &= \sqrt{2} \\
c &= 1
\end{align*} \]  \hspace{1cm} (26-28)

Normalizing the eigenvector gives
The other two eigenvectors can be found the same way, with the result

\[
|L_x = +1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

(29)

\[
|L_x = 0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

(30)

\[
|L_x = -1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix}
\]

(31)

Note that these eigenvectors are orthonormal.

Now that we have the eigenvectors of \(L_x\) we can answer the following question. If we start with the state \(|L_z = -1\rangle\) and measure \(L_x\), what are the possible outcomes and the probability of each?

First, we need to express \(|L_z = -1\rangle\) in terms of the eigenvectors of \(L_x\) which we can do by solving three simultaneous linear equations, and we find

\[
|L_z = -1\rangle = \frac{1}{\sqrt{2}} (|L_x = +1\rangle + |L_x = -1\rangle) - \frac{1}{\sqrt{2}} |L_x = 0\rangle
\]

(32)

(You can verify this by direct substitution.) Thus all 3 possible values of \(L_x\) can result from a measurement, and the probability of each is
Now suppose we start with the state, written in the $L_z$ basis:

$$|\psi\rangle = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$  \hspace{1cm} (42)$$

We take a measurement of $L_z^2$ and obtain $+1$. The operator $L_z^2$ is given by squaring $5$:

$$L_z^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (43)$$

This has a degenerate eigenvalue $\lambda = +1$, so the most we can say about the state $|\psi\rangle$ after the measurement is that it is projected onto the subspace

$$a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  \hspace{1cm} That is, the state after the measurement is given by
\[ |\psi\rangle_{after} = \mathbb{P}_{L_z = \pm 1} |\psi\rangle_{before} \tag{44} \]

\[ = [|L_z = +1\rangle \langle L_z = +1| + |L_z = -1\rangle \langle L_z = -1|] \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{45} \]

\[ = \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{46} \]

\[ = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{47} \]

We can normalize this state to get

\[ |\psi\rangle_{after} = \frac{2}{\sqrt{3}} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{48} \]

Thus if we measure \( L_z \) immediately after the measurement of \( L_z \) above, we get \( L_z = +1 \) with probability \( \frac{1}{3} \) and \( L_z = -1 \) with probability \( \frac{2}{3} \).

Finally, suppose we have a state \( |\psi\rangle \) with the probabilities of measurements of \( L_z \) given as \( P(L_z = 1) = \frac{1}{4} \), \( P(L_z = 0) = \frac{1}{2} \) and \( P(L_z = -1) = \frac{1}{4} \). Since these probabilities are given by \( |\langle L_z = \lambda |\psi\rangle|^2 \) for each of the three possible values of \( \lambda \), and the vectors \( |L_z = \lambda\rangle \) are orthonormal, the most general form for \( |\psi\rangle \) is

\[ |\psi\rangle = \frac{e^{i\delta_1}}{2} |L_z = 1\rangle + \frac{e^{i\delta_2}}{\sqrt{2}} |L_z = 0\rangle + \frac{e^{i\delta_3}}{2} |L_z = -1\rangle \tag{49} \]

where the \( \delta_i \) are real numbers. For example

\[ |\langle L_z = 1 |\psi\rangle|^2 = \left| \frac{e^{i\delta_1}}{2} \right|^2 = \frac{1}{4} \tag{50} \]

While the presence of a phase factor in a solitary state doesn’t affect the physics of that state, if we have a sum of states, each with its own (different) phase factor, we can’t ignore these phase factors. For example, if we measure \( L_x \) in this state and want the probability that \( L_x = 0 \), we have, using [30]...
The last line will have a different result for different values of the phase factors $\delta_1$ and $\delta_3$, so they can't be ignored.