

PROBABILITY CURRENT

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Using the Schrödinger equation we can derive an interesting quantity called the *probability current*. Using the probabilistic interpretation of the wave function, the probability of a particle being between $x = a$ and $x = b$ is

$$P_{ab} = \int_a^b |\Psi^* \Psi| dx \quad (1)$$

The rate of change of this probability can then be expressed in terms of spatial derivatives using the Schrödinger equation:

$$\frac{dP_{ab}}{dt} = \int_a^b \left[\frac{\partial \Psi^*}{\partial t} \Psi + \frac{\partial \Psi}{\partial t} \Psi^* \right] dx \quad (2)$$

$$= \int_a^b \left\{ -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} - \frac{1}{i\hbar} V \Psi^* \right\} \Psi dx \quad (3)$$

$$+ \int_a^b \left\{ \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{i\hbar} V \Psi \right\} \Psi^* dx \quad (4)$$

$$= \frac{i\hbar}{2m} \int_a^b \left[\frac{\partial^2 \Psi}{\partial x^2} \Psi^* - \frac{\partial^2 \Psi^*}{\partial x^2} \Psi \right] dx \quad (5)$$

We can now apply integration by parts to each term.

$$\int_a^b \frac{\partial^2 \Psi}{\partial x^2} \Psi^* dx = \frac{\partial \Psi}{\partial x} \Psi^* \Big|_a^b - \int_a^b \frac{\partial \Psi}{\partial x} \frac{\partial \Psi^*}{\partial x} dx \quad (6)$$

$$- \int_a^b \frac{\partial^2 \Psi^*}{\partial x^2} \Psi dx = -\frac{\partial \Psi^*}{\partial x} \Psi \Big|_a^b + \int_a^b \frac{\partial \Psi}{\partial x} \frac{\partial \Psi^*}{\partial x} dx \quad (7)$$

Adding these terms together, we get

$$\frac{dP_{ab}}{dt} = \frac{i\hbar}{2m} \left[\frac{\partial \Psi}{\partial x} \Psi^* \Big|_a^b - \frac{\partial \Psi^*}{\partial x} \Psi \Big|_a^b \right] \quad (8)$$

If we define the *probability current* as

$$J(x,t) \equiv \frac{i\hbar}{2m} \left(\frac{\partial \Psi^*}{\partial x} \Psi - \frac{\partial \Psi}{\partial x} \Psi^* \right) \quad (9)$$

we can write the rate of change of probability as

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t) \quad (10)$$

We can interpret this as a probability conservation law. The rate of change of probability in the spatial interval $x \in [a, b]$ is the difference between the probability current flowing in at one end ($x = a$, say) and out at the other ($x = b$).

As an example, if the wave function is given by

$$\Psi(x,t) = \left(\frac{2am}{\pi\hbar} \right)^{1/4} e^{-a[(mx^2/\hbar)+it]} \quad (11)$$

(we've taken the constant in front so that it normalizes the wave function), then

$$\frac{\partial \Psi}{\partial x} = -\frac{1}{\pi^{1/4}} \left(\frac{2am}{\hbar} \right)^{5/4} x e^{-a[(mx^2/\hbar)+it]} \quad (12)$$

So for the probability current we get

$$J(x,t) = -\frac{i\hbar}{2m} \frac{1}{\pi^{1/4}} \left(\frac{2am}{\hbar} \right)^{5/4} x e^{-a[(mx^2/\hbar)-it]} \left(\frac{2am}{\pi\hbar} \right)^{1/4} e^{-a[(mx^2/\hbar)+it]} \quad (13)$$

$$+ \frac{i\hbar}{2m} \frac{1}{\pi^{1/4}} \left(\frac{2am}{\hbar} \right)^{5/4} x e^{-a[(mx^2/\hbar)+it]} \left(\frac{2am}{\pi\hbar} \right)^{1/4} e^{-a[(mx^2/\hbar)-it]} \quad (14)$$

$$= 0 \quad (15)$$

A bit of an anti-climax after all that mathematics.

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