PROJECTION OPERATORS

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Post date: 26 July 2021.

We've been writing vector inner products using the Dirac bra-ket notation, so the inner product of two vectors is $\langle f|g\rangle$. Dirac's idea was to break this notation into two pieces, the 'bra' and the 'ket'. The meaning of the ket part is fairly obvious: it's just the original vector. But what exactly is the 'bra' part? Essentially, it's a linear operator whose operand is a vector and output is a complex number (scalar). If the vector space is discrete (containing any number of dimensions, finite or infinite), then applying a bra to a ket results in the ordinary scalar product (the 'dot product' familiar from linear algebra). If the vector space is continuous, as with position or momentum, then applying a bra to a ket results in an integral over the relevant domain.

It's worth pointing out that some authors such as Griffiths call the bra a linear function of vectors rather than an operator, preferring to reserve the term 'operator' for something which operates on a vector and returns another vector. I don't see any particular value in such a fine distinction, and since the bra certainly does 'operate' on a vector (even though it produces a scalar as the result), the term 'operator' seems appropriate.

Although the bra has no physical meaning on its own, it can still simplify the notation for some other operators. One such example is the *projection operator*. If you can remember your linear algebra, you might recall that, given two vectors **a** and **b**, you can find the perpendicular projection of **a** on **b** from the formula

$$\mathbf{a}_{\perp}=\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^{2}}\mathbf{b} \tag{1}$$

If **b** is normalized (that is, it's a unit vector), then this formula reduces to

$$\mathbf{a}_{\perp}=(\mathbf{b}\cdot\mathbf{a})\,\mathbf{b} \tag{2}$$

This amounts to taking the inner (dot) product of **b** with **a** and multiplying by the vector **b**. That is, we take a bra of **b** and have it operate on a ket of **a**, then multiply the result into the ket of **b**.

In bra-ket notation, we can define the projection operator as

$$\hat{P} \equiv |\alpha\rangle\langle\alpha| \tag{3}$$

where $|\alpha\rangle$ is a normalized vector. Applying this to any other vector $|\beta\rangle$ gives the projection of $|\beta\rangle$ along $|\alpha\rangle$:

$$|\beta\rangle_{\perp} = |\alpha\rangle\langle\alpha|\beta\rangle \tag{4}$$

We'll have a look at a few properties of the projection operator.

First, the projection operator is *idempotent*, which means that $\hat{P}^2 = \hat{P}$. The consequence of this is that it doesn't matter how many times you apply a given projection operator; it will have the same result as applying it just once. This makes sense from a geometric viewpoint, since once you've projected a vector onto another vector, projecting the projection just gives you the same projection back again.

The proof of the idempotent property is quite simple: $\hat{P}^2 = |\alpha\rangle\langle\alpha|\alpha\rangle\langle\alpha| = |\alpha\rangle\langle\alpha|$, since $\langle\alpha|\alpha\rangle = 1$.

Since it's an operator that returns a vector, we can find its eigenvalues. Using the idempotent property we get

$$\hat{P}^2|a\rangle = \hat{P}|a\rangle \tag{5}$$

$$(\hat{P}^2 - \hat{P})|a\rangle = 0 \tag{6}$$

$$(p^2 - p)|a\rangle = 0 \tag{7}$$

where p is an eigenvalue of $\hat{P}$. Thus the only two eigenvalues possible are 0 and 1.

For an eigenvalue of 1, the corresponding eigenvector must satisfy $\hat{P}|a\rangle = |\alpha\rangle\langle\alpha|a\rangle = |a\rangle$. Thus the eigenvector for eigenvalue 1 is $A|\alpha\rangle$ for some constant A. Thus any vector parallel to $|\alpha\rangle$ is an eigenvector.

For an eigenvalue of 0, we have $\hat{P}|a\rangle = |\alpha\rangle\langle\alpha|a\rangle = 0$, so the eigenvector is any vector orthogonal to $|\alpha\rangle$.