QUANTUM REVIVAL TIME IN INFINITE SQUARE WELL

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Reference: Styer, Daniel F. (2000), Quantum revivals versus classical periodicity in the infinite square well. (Online paper).

Suppose a particle is in an infinite square well in the interval $x \in [0, a]$. We're not specifying which state it's in, so in general, its wave function is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar} \tag{1}$$

where the coefficients c_n would be determined in the usual way by looking at $\Psi(x,0)$.

Any wave function in the infinite square well has what is known as a *revival time*, which is a time T after which the wave function returns to its initial state. We can find this time by looking at the time-dependent term above.

$$\Psi(x,t+T) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n (t+T)/\hbar} \tag{2}$$

In order for this to be equal to $\Psi(x,t)$, we must have

$$e^{-iE_n (t+T)/\hbar} = e^{-iE_n t/\hbar} \tag{3}$$

The first T for which this is true is found from

$$\frac{E_n (t+T)}{\hbar} = \frac{E_n t}{\hbar} + 2\pi \tag{4}$$

$$T = \frac{2\pi\hbar}{E_n} \tag{5}$$

The infinite square well energies are

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} \tag{6}$$

so

$$T = \frac{4ma^2}{n^2\pi\hbar} \tag{7}$$

We want a value of T which will result in a return to the initial state for *all* energies, so we need to pick the longest time, which occurs at $n = 1$. This value of T is a multiple of all the times calculated using higher values of n, so the argument of the complex exponential will be a multiple of 2π for all these times, ensuring a return to the initial state. Thus the revival time is

$$T = \frac{4ma^2}{\pi\hbar} \tag{8}$$

Classically, a particle in an infinite square well bounces back and forth between the walls. If its energy is $E_c = mv^2/2$, its velocity is $v = \sqrt{2E_c/m}$ and the classical revival time is the time taken to traverse the well twice:

$$T_c = \frac{2a}{v} \tag{9}$$

$$= \sqrt{\frac{2m}{E_c}}a \tag{10}$$

This is equal to the quantum time if

$$T = T_c \tag{11}$$

$$\frac{4ma^2}{\pi\hbar} = \sqrt{\frac{2m}{E_c}}a \tag{12}$$

$$E_c = \frac{\pi^2\hbar^2}{8ma^2} \tag{13}$$

Since this is smaller than the ground state energy in the quantum system, this energy cannot be realized in the quantum system.

In fact, there seems to be a greater problem than this: the classical revival time depends on the energy, while the quantum time does not; it depends only on the particle's mass and the width of the well. In Styer's paper (reference at the top), he argues that the full wave function is not measurable, so what we should be looking at are measurable quantities, such as the expectation value of position $\langle x\rangle$. He shows that for a particle made up of a superposition of stationary states centred at quantum number n_C, the period of $\langle x\rangle$ is

$$T_{\langle x\rangle} = \frac{T}{2n_C} \tag{14}$$

Given that the quantum energy of state n_C is $E_C = E_1 n_C^2$, we can write this as

$$T_{\langle x\rangle} = \frac{T}{2\sqrt{E_C/E_1}} \tag{15}$$

$$= \frac{2\pi\hbar}{2E_1\sqrt{E_C/E_1}} \tag{16}$$

$$= \frac{\pi\hbar}{\sqrt{E_C E_1}} \tag{17}$$

Using $E_1 = \pi^2\hbar^2/2ma^2$ we get

$$T_{\langle x\rangle} = \sqrt{\frac{2m}{E_C}}a \tag{18}$$

That is, the revival time for the position in the quantum case *is* equal to the classical revival time.