REFLECTIONLESS POTENTIAL

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An interesting potential is

$$V(x) = -\frac{\hbar^2 a^2}{m}\text{sech}^2(ax) \tag{1}$$

where a is a constant and $\text{sech}(ax)$ is the hyperbolic secant, which is defined as $\text{sech}(ax) \equiv 1/\cosh(ax)$. The general shape of this potential is as shown in Fig. 1.

We can verify by direct substitution that the function

$$\psi_0(x) = A\text{sech}(ax) \tag{2}$$

is a solution. We use the derivatives

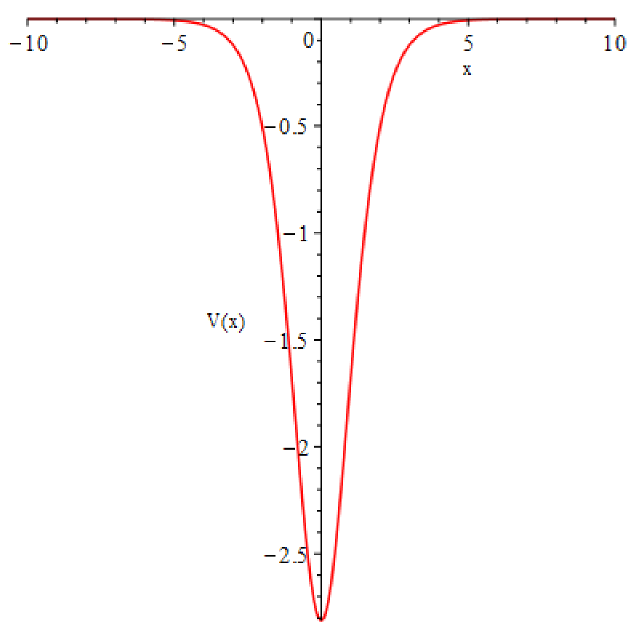

FIGURE 1. Reflectionless potential.

$$\begin{aligned}\frac{d}{dx}\text{sech}x &= -\tanh x\,\text{sech}x \\ \frac{d}{dx}\tanh x &= 1-\tanh^2 x\end{aligned} \tag{3}$$

We get

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_0}{dx^2}+V\psi_0 = -\frac{\hbar^2}{2m}Aa^2\text{sech}\,(ax) \tag{4}$$

$$= -\frac{\hbar^2 a^2}{2m}\psi_0 \tag{5}$$

Thus the energy of this state is

$$E_0 = -\frac{\hbar^2 a^2}{2m} \tag{6}$$

We can normalize ψ_0 to find A:

$$\int_{-\infty}^{\infty}\psi_0^2 dx = 1 \tag{7}$$

$$A = \sqrt{a/2} \tag{8}$$

so we have

$$\psi_0\,(x) = \sqrt{\frac{a}{2}}\text{sech}\,(ax) \tag{9}$$

A plot of $\psi_0(x)$ with $a = 0.75$ is in Fig. 2.

For positive energies, we can verify that

$$\psi_k(x) = B\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx} \tag{10}$$

is a solution of the Schrodinger equation for any energy by direct substitution. Here, as usual, $k \equiv \sqrt{2mE}/\hbar$.

We get

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_k}{dx^2}+V\psi_k = \frac{\hbar^2 k^2 B}{2m}\left(\frac{ik - a\tanh\,(ax)}{ik+a}\right) \tag{11}$$

$$= \frac{\hbar^2 k^2}{2m}\psi_k \tag{12}$$

$$= E\psi_k \tag{13}$$

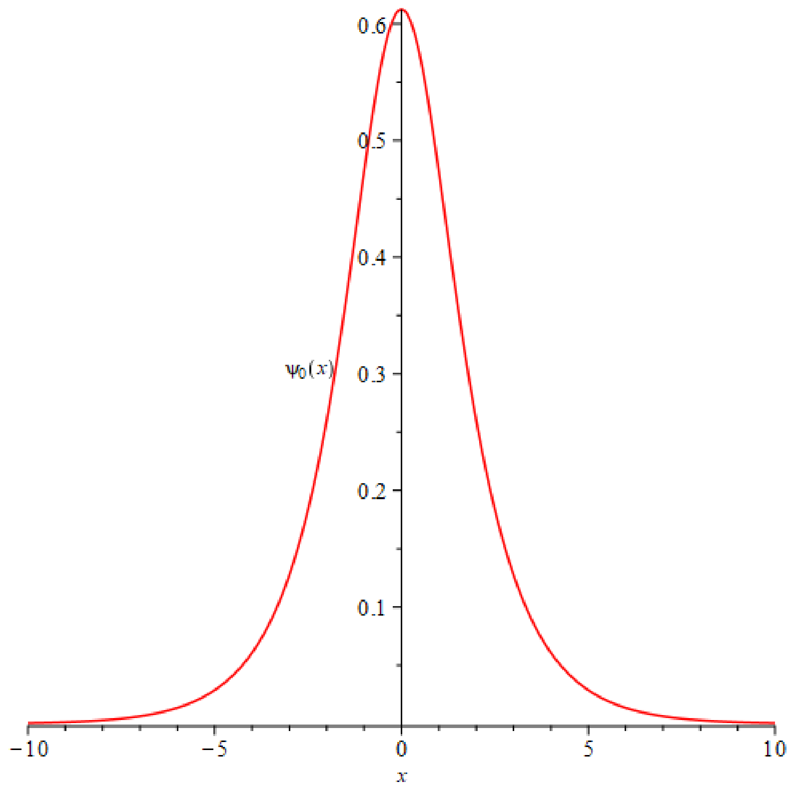

FIGURE 2. Wave function ψ_0.

The asymptotic behaviour of ψ_k can be found from the limit $\lim_{x\to\infty}\tanh(ax) = 1$. We therefore get:

$$\lim_{x\to\infty}\psi_k(x) = B\frac{ik-a}{ik+a}e^{ikx} \tag{14}$$

$$= -B\frac{(ik-a)^2}{a^2+k^2}e^{ikx} \tag{15}$$

For large negative x $\lim_{x\to-\infty}\tanh(ax) = -1$ so we get

$$\lim_{x\to-\infty}\psi_k(x) = B\frac{ik+a}{ik+a}e^{ikx} \tag{16}$$

$$= Be^{ikx} \tag{17}$$

Thus in both cases, the wave function represents a wave travelling to the right, with no leftward component. That is, there is no reflected wave. The modulus of the wave for large x is

$$\lim_{x\to\infty}\left|\psi_k(x)\right|^2 = |B|^2\left|\frac{(ik-a)^2}{a^2+k^2}\right|^2 \tag{18}$$

$$= |B|^2 \tag{19}$$

$$= \lim_{x\to-\infty}\left|\psi_k(x)\right|^2 \tag{20}$$

Thus the transmission coefficient is 1 for all positive energies, which means that any particle coming in from the left passes straight through with no reflection. There is, however, a change of phase due to the factor of $\frac{(ik-a)^2}{a^2+k^2}$.