

ROTATION OF A VECTOR WAVE FUNCTION

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We've seen that, for a rotation by an infinitesimal angle ε_z about the z axis, a scalar wave function transforms according to

$$\psi(x, y) \rightarrow \psi(x + \varepsilon_z y, y - \varepsilon_z x) \quad (1)$$

The meaning of this transformation can be seen in Fig. 1.

The physical system represented by the wave function Ψ is rigidly rotated by the angle ε_z , so that the value of Ψ at point A is now sitting over the point B . However, in the primed (rotated) coordinate system, the numerical value of the coordinates of the point B in the figure are the same as the numerical values that the point A had in the original, unrotated coordinates. That is

$$(x'_B, y'_B) = (x_A, y_A) \quad (2)$$

Just as B is obtained from A by rotating A by $+\varepsilon_z$, we can obtain A from B by rotating by $-\varepsilon_z$. For any given point, the primed (rotated) and unprimed (unrotated) coordinates are related by (all relations are to first order in ε_z):

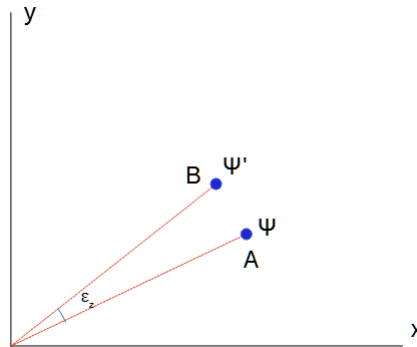
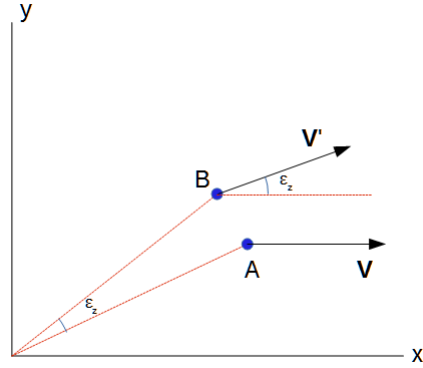


FIGURE 1. Rotation of a scalar wave function Ψ .


 FIGURE 2. Rotation of a vector wave function \mathbf{V} .

$$x' = x - y\varepsilon_z \quad (3)$$

$$y' = y + x\varepsilon_z \quad (4)$$

The inverse relations are obtained by a rotation by $-\varepsilon_z$:

$$x = x' + y'\varepsilon_z \quad (5)$$

$$y = y' - x'\varepsilon_z \quad (6)$$

After rotation, the values of Ψ' are related to the values Ψ before rotation by rotating through the angle $-\varepsilon_z$, so that

$$\Psi'(x, y) = \Psi(x + y\varepsilon_z, y - x\varepsilon_z) \quad (7)$$

Now suppose the wave function is a vector $\mathbf{V} = V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}}$ in two dimensions. The situation is as shown in Fig. 2.

The initial unrotated vector \mathbf{V} is the value of the wave function at point A (and is entirely in the x direction for convenience). After rotation, the vector gets moved to B and is also rotated so that it now makes an angle ε_z with the *original* x axis. However, its direction is now along the x' axis, which makes an angle of ε_z with the original x axis.

In this case, each component of \mathbf{V} still gets transformed in the same way as the scalar function above, but the vector itself is also rotated. If the components V_x and V_y of the vector are constants over all space, then the rotated vector is given by applying the 2-d rotation matrix

$$R = \begin{bmatrix} 1 & -\varepsilon_z \\ \varepsilon_z & 1 \end{bmatrix} \quad (8)$$

so we get $\mathbf{V}' = R\mathbf{V}$, or, in components:

$$V'_x = V_x - V_y \varepsilon_z \quad (9)$$

$$V'_y = V_y + V_x \varepsilon_z \quad (10)$$

If V_x and V_y vary from point to point, then we must apply the transformation 1 to each component, so that the overall transformation is

$$V'_x = V_x(x + \varepsilon_z y, y - \varepsilon_z x) - V_y(x + \varepsilon_z y, y - \varepsilon_z x) \varepsilon_z \quad (11)$$

$$V'_y = V_y(x + \varepsilon_z y, y - \varepsilon_z x) + V_x(x + \varepsilon_z y, y - \varepsilon_z x) \varepsilon_z \quad (12)$$

In terms of the rotation matrix R , the overall transformation can be written as

$$\mathbf{V}' = R\mathbf{V}(R^{-1}\mathbf{r}) \quad (13)$$

where

$$\mathbf{r} = (x, y) \quad (14)$$

is the position vector. That is, we apply the *inverse* rotation to the coordinates and the *ordinary* rotation to the vector.

The operator that generates the transformation of a scalar function by an infinitesimal angle $\delta\theta$ is

$$U[R(\delta\theta)] = I - \frac{i}{\hbar} \delta\theta \cdot \mathbf{L} \quad (15)$$

In this case, the rotation is about the z axis so

$$\delta\theta = \varepsilon_z \hat{\mathbf{z}} \quad (16)$$

$$\delta\theta \cdot \mathbf{L} = \varepsilon_z L_z \quad (17)$$

Thus we have

$$V_{x,y}(x + \varepsilon_z y, y - \varepsilon_z x) = \left(I - \frac{i}{\hbar} \varepsilon_z L_z \right) V_{x,y}(x, y) \quad (18)$$

Plugging this into 11 and keeping terms only up to order ε_z we have

$$V'_x = \left(I - \frac{i}{\hbar} \varepsilon_z L_z \right) V_x - V_y \varepsilon_z \quad (19)$$

$$V'_y = \left(I - \frac{i}{\hbar} \varepsilon_z L_z \right) V_y + V_x \varepsilon_z \quad (20)$$

In matrix form, this is

$$\begin{bmatrix} V'_x \\ V'_y \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i\varepsilon_z}{\hbar} \begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} - \varepsilon_z \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad (21)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i\varepsilon_z}{\hbar} \begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} - \frac{i\varepsilon_z}{\hbar} \begin{bmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{bmatrix} \right) \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad (22)$$

$$\equiv \left(I - \frac{i\varepsilon_z}{\hbar} J_z \right) \begin{bmatrix} V_x \\ V_y \end{bmatrix} \quad (23)$$

This has the same form as 15, except that the angular momentum generator is now the sum of L_z and the final matrix on the RHS above, which Shankar calls suggestively S_z , in anticipation of spin which at this stage he hasn't considered. That is,

$$J_z = L_z + S_z \quad (24)$$

$$= \begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix} + \begin{bmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{bmatrix} \quad (25)$$

The eigenvalues of the second matrix here are just $\pm\hbar$, so we haven't yet encountered half-integral values of angular momentum.