

ROTATIONAL TRANSFORMATIONS USING PASSIVE TRANSFORMATIONS

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Post date: 22 July 2021.

We can also derive the generator of rotations L_z by considering passive transformations of the position and momentum operators, in a way similar to that used for deriving the generator of translations. In a passive transformation, the operators are modified while the state vectors remain the same. For an infinitesimal rotation $\varepsilon_z \hat{\mathbf{z}}$ about the z axis in two dimensions, the unitary operator has the form

$$U[R(\varepsilon_z \hat{\mathbf{z}})] = I - \frac{i\varepsilon_z L_z}{\hbar} \quad (1)$$

For a finite rotation by $\phi_0 \hat{\mathbf{z}}$ the transformations are given by

$$\langle X \rangle_R = \langle X \rangle \cos \phi_0 - \langle Y \rangle \sin \phi_0 \quad (2)$$

$$\langle Y \rangle_R = \langle X \rangle \sin \phi_0 + \langle Y \rangle \cos \phi_0 \quad (3)$$

$$\langle P_x \rangle_R = \langle P_x \rangle \cos \phi_0 - \langle P_y \rangle \sin \phi_0 \quad (4)$$

$$\langle P_y \rangle_R = \langle P_x \rangle \sin \phi_0 + \langle P_y \rangle \cos \phi_0 \quad (5)$$

For the infinitesimal transformation, $\phi_0 = \varepsilon_z$ and these equations reduce to

$$\langle X \rangle_R = \langle X \rangle - \langle Y \rangle \varepsilon_z \quad (6)$$

$$\langle Y \rangle_R = \langle X \rangle \varepsilon_z + \langle Y \rangle \quad (7)$$

$$\langle P_x \rangle_R = \langle P_x \rangle - \langle P_y \rangle \varepsilon_z \quad (8)$$

$$\langle P_y \rangle_R = \langle P_x \rangle \varepsilon_z + \langle P_y \rangle \quad (9)$$

In the passive transformation scheme, we move the transformation to the operators to get

$$U^\dagger [R] X U [R] = X - Y \varepsilon_z \quad (10)$$

$$U^\dagger [R] Y U [R] = X \varepsilon_z + Y \quad (11)$$

$$U^\dagger [R] P_x U [R] = P_x - P_y \varepsilon_z \quad (12)$$

$$U^\dagger [R] P_y U [R] = P_x \varepsilon_z + P_y \quad (13)$$

Substituting 1 into these equations gives us the commutation relations satisfied by L_z . For example, in the first equation we have

$$U^\dagger [R] X U [R] = \left(I + \frac{i\varepsilon_z L_z}{\hbar} \right) X \left(I - \frac{i\varepsilon_z L_z}{\hbar} \right) \quad (14)$$

$$= X + \frac{i\varepsilon_z}{\hbar} (L_z X - X L_z) \quad (15)$$

$$= X - Y \varepsilon_z \quad (16)$$

Equating the last two lines, we get

$$[X, L_z] = -i\hbar Y \quad (17)$$

Similarly, for the other three equations we get

$$[Y, L_z] = i\hbar X \quad (18)$$

$$[P_x, L_z] = -i\hbar P_y \quad (19)$$

$$[P_y, L_z] = i\hbar P_x \quad (20)$$

We can use these commutation relations to derive the form of L_z by using the commutation relations for coordinates and momenta:

$$[X, P_x] = [Y, P_y] = i\hbar \quad (21)$$

with all other commutators involving X, Y, P_x and P_y being zero. Starting with 17, we see that

$$[X, L_z] = -[X, P_x] Y \quad (22)$$

We can therefore deduce that

$$L_z = -P_x Y + f(X, Y, P_y) \quad (23)$$

where f is some unknown function. We must include f since the commutators of X with X, Y and P_y are all zero, so adding on f still satisfies 17. (You can think of it as similar to adding on the constant in an indefinite integral.)

Now from 18, we have

$$[Y, L_z] = [Y, P_y] X \quad (24)$$

so combining this with 23 we have

$$L_z = -P_x Y + P_y X + g(X, Y) \quad (25)$$

The undetermined function is now a function only of X and Y , since the dependence of L_z on P_x and P_y has been determined uniquely by the commutators 17 and 18.

From 19 we have

$$[P_x, L_z] = [P_x, X] P_y \quad (26)$$

We can see that this is satisfied already by 25, except that we now know that the function g cannot depend on X , since then $[P_x, g] \neq 0$. Thus we have narrowed down L_z to

$$L_z = -P_x Y + P_y X + h(Y) \quad (27)$$

Finally, from 20 we have

$$[P_y, L_z] = -[P_y, Y] P_x \quad (28)$$

This is satisfied by 27 if we take $h = 0$ (well, technically, we could take h to be some constant, but we might as well take the constant to be zero), giving us the final form for L_z :

$$L_z = -P_x Y + P_y X \quad (29)$$

PINGBACKS

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