

ROTATIONS THROUGH A FINITE ANGLE - USE OF POLAR COORDINATES

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The angular momentum operator L_z is the generator of rotations in the xy plane. We did the derivation for infinitesimal rotations, but we can generalize this to finite rotations in a similar manner to that used for translations. The unitary transformation for an infinitesimal rotation is

$$U[R(\varepsilon_z \hat{\mathbf{z}})] = I - \frac{i\varepsilon_z L_z}{\hbar} \quad (1)$$

For rotation through a finite angle ϕ_0 , we divide up the angle into N small angles, so $\varepsilon_z = \phi_0/N$. Rotation through the full angle ϕ_0 is then given by

$$U[R(\phi_0 \hat{\mathbf{z}})] = \lim_{N \rightarrow \infty} \left(I - \frac{i\phi_0 L_z}{N\hbar} \right)^N = e^{-i\phi_0 L_z/\hbar} \quad (2)$$

The limit follows because the only non-trivial operator involved is L_z , so no commutation problems arise.

In rectangular coordinates, L_z has the relatively non-obvious form

$$L_z = X P_y - Y P_x \quad (3)$$

$$= -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \quad (4)$$

so it's not immediately clear that 2 does in fact lead to the desired rotation. Trying to calculate the exponential with L_z expressed this way is not easy, given that the two terms $x \frac{\partial}{\partial y}$ and $y \frac{\partial}{\partial x}$ don't commute.

It turns out that L_z has a much simpler form in polar coordinates, and there are two ways of converting it to polar form. First, we recall the transformation equations.

$$x = \rho \cos \phi \quad (5)$$

$$y = \rho \sin \phi \quad (6)$$

$$\rho = \sqrt{x^2 + y^2} \quad (7)$$

$$\phi = \tan^{-1} \frac{y}{x} \quad (8)$$

From the chain rule, we can convert the derivatives:

$$\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \cos \phi}{\partial x} \frac{\partial}{\partial (\cos \phi)} \quad (9)$$

$$= \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} - \sin \phi \frac{\partial \phi}{\partial x} \frac{\partial}{\partial (-\sin \phi)} \quad (10)$$

$$= \frac{x}{\rho} \frac{\partial}{\partial \rho} - \sin \phi \frac{-y/x^2}{1 + y^2/x^2} \left(\frac{-1}{\sin \phi} \right) \frac{\partial}{\partial \phi} \quad (11)$$

$$= \frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi} \quad (12)$$

Using similar methods, we get for the other derivative

$$\frac{\partial}{\partial y} = \frac{\partial \rho}{\partial y} \frac{\partial}{\partial \rho} + \frac{\partial \sin \phi}{\partial y} \frac{\partial}{\partial (\sin \phi)} \quad (13)$$

$$= \frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi} \quad (14)$$

Plugging these into 4 we have

$$L_z = -i\hbar \left[x \left(\frac{y}{\rho} \frac{\partial}{\partial \rho} + \frac{x}{\rho^2} \frac{\partial}{\partial \phi} \right) - y \left(\frac{x}{\rho} \frac{\partial}{\partial \rho} - \frac{y}{\rho^2} \frac{\partial}{\partial \phi} \right) \right] \quad (15)$$

$$= -i\hbar \frac{x^2 + y^2}{\rho^2} \frac{\partial}{\partial \phi} \quad (16)$$

$$= -i\hbar \frac{\partial}{\partial \phi} \quad (17)$$

Another method of converting L_z to polar coordinates is to consider the effect of $U[R]$ for an infinitesimal rotation ε_z on a state vector expressed in polar coordinates $\psi(\rho, \phi)$. Shankar states that

$$\langle \rho, \phi | U[R] | \psi(\rho, \phi) \rangle = \psi(\rho, \phi - \varepsilon_z) \quad (18)$$

If you don't believe this, it can be shown using a method similar to that for the one-dimensional translation. In this case, we're dealing with position eigenkets in polar coordinates, so we have

$$U[R]|\rho, \phi\rangle = |\rho, \phi + \varepsilon_z\rangle \quad (19)$$

Applying this, we get

$$|\psi_{\varepsilon_z}\rangle = U[R]|\psi\rangle \quad (20)$$

$$= U[R] \int_0^{2\pi} \int_0^\infty |\rho, \phi\rangle \langle \rho, \phi | \psi \rangle \rho d\rho d\phi \quad (21)$$

$$= \int_0^{2\pi} \int_0^\infty |\rho, \phi + \varepsilon_z\rangle \langle \rho, \phi | \psi \rangle \rho d\rho d\phi \quad (22)$$

$$= \int_0^{2\pi} \int_0^\infty |\rho', \phi'\rangle \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi' \quad (23)$$

where in the last line, we used the substitution $\phi' = \phi + \varepsilon_z$. (The substitution $\rho' = \rho$ is used just to give the radial variable a different name in the integrand.) We can use the same limits of integration for ϕ and ϕ' , since we just need to ensure that the integral covers the total range of angles. It then follows that

$$\langle \rho, \phi | \psi_{\varepsilon_z} \rangle = \int_0^{2\pi} \int_0^\infty \langle \rho, \phi | \rho', \phi' \rangle \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi' \quad (24)$$

$$= \int_0^{2\pi} \int_0^\infty \delta(\rho - \rho') \delta(\phi - \phi') \langle \rho', \phi' - \varepsilon_z | \psi \rangle \rho' d\rho' d\phi' \quad (25)$$

$$= \psi(\rho, \phi - \varepsilon_z) \quad (26)$$

Combining this with 1 we have

$$\left\langle \rho, \phi \left| I - \frac{i\varepsilon_z L_z}{\hbar} \right| \psi \right\rangle = \psi(\rho, \phi - \varepsilon_z) \quad (27)$$

Expanding the RHS to order ε_z we have

$$\left\langle \rho, \phi \left| I - \frac{i\varepsilon_z L_z}{\hbar} \right| \psi \right\rangle = \psi(\rho, \phi) - \varepsilon_z \frac{\partial \psi}{\partial \phi} \quad (28)$$

from which 17 follows again.

Once we have L_z in this form, the exponential form of a finite rotation is easier to interpret, for we have, from 2

$$e^{-i\phi_0 L_z/\hbar} = \exp\left[-\phi_0 \frac{\partial}{\partial \phi}\right] \quad (29)$$

$$= 1 - \phi_0 \frac{\partial}{\partial \phi} + \frac{\phi_0^2}{2!} \frac{\partial^2}{\partial \phi^2} + \dots \quad (30)$$

Applying this to a state function $\psi(\rho, \phi)$, we see that we get the Taylor series for $\psi(\rho, \phi - \phi_0)$, so the exponential does indeed represent a rotation through a finite angle.

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