SPECTRAL DECOMPOSITION OF OPERATORS

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Suppose we have an operator $\hat{Q}$ with a complete, orthonormal set of eigenvectors, so that

$$\hat{Q}|e_m\rangle = q_m|e_m\rangle \tag{1}$$

where q_m is the eigenvalue. We can write the operator as a *spectral decomposition* operator, as follows.

Since the eigenvectors of $\hat{Q}$ form a complete, orthonormal set, we can write any vector $|\alpha\rangle$ in terms of that set:

$$|\alpha\rangle = \sum_m a_m|e_m\rangle \tag{2}$$

where a_m is the coefficient of the basis vector $|e_m\rangle$. Applying $\hat{Q}$ directly to this expansion gives:

$$\hat{Q}|\alpha\rangle = \sum_m a_m q_m|e_m\rangle \tag{3}$$

However, if we write $\hat{Q}$ as

$$\hat{Q} = \sum_n q_n|e_n\rangle\langle e_n| \tag{4}$$

we get

$$\hat{Q}|\alpha\rangle = \left[\sum_n q_n|e_n\rangle\langle e_n|\right]\sum_m a_m|e_m\rangle \tag{5}$$

$$= \sum_{n,m} q_n a_m \delta_{nm}|e_n\rangle\langle e_m| \tag{6}$$

$$= \sum_m a_m q_m|e_m\rangle \tag{7}$$

where the result follows from the orthonormality of the set of eigenvectors: $\langle e_m|e_n\rangle = \delta_{mn}$. Since this spectral decomposition form of $\hat{Q}$ has the same action on any vector as its original form, the two forms must be equivalent.