

SPHERICAL HARMONICS - EXAMPLES

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Post date: 30 July 2021.

In the solution of the 3-d Schrödinger equation with a radial potential, we found that the angular part could be written in terms of spherical harmonics

$$Y_l^m(\theta, \phi) = \epsilon \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} e^{im\phi} P_l^m(\cos\theta) \quad (1)$$

with $\epsilon = (-1)^m$ for $m > 0$ and $\epsilon = 1$ for $m < 0$ and the P_l^m being the associated Legendre function:

$$P_l^m(x) = (1-x^2)^{m/2} \sum_{k=0}^{[(l-m)/2]} \frac{(2l-2k)!}{2^l(l-k)!k!(l-2k-m)!} (-1)^k x^{l-m-2k} \quad (2)$$

As an example, we can calculate a couple of the spherical harmonics from these formulas.

Y_0^0 can be calculated easily from inspection of the formulas and comes out to

$$Y_0^0 = \left(\frac{1}{4\pi} \right)^{1/2} \quad (3)$$

To get Y_2^1 , we start with 1:

$$Y_2^1 = -\sqrt{\frac{5}{24\pi}} e^{i\phi} P_2^1(\cos\theta) \quad (4)$$

Then from 2

$$P_2^1(x) = (1-x^2)^{1/2} \sum_{k=0}^{[1/2]} \frac{(4-2k)!}{2^2(2-k)!k!(2-2k-1)!} (-1)^k x^{1-2k} \quad (5)$$

$$= (1-x^2)^{1/2} (3x) \quad (6)$$

$$P_2^1(\cos\theta) = 3 \sin\theta \cos\theta \quad (7)$$

So finally we get

$$Y_2^1(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} e^{i\phi} \sin\theta \cos\theta \quad (8)$$

The normalization can be checked by direct integration

$$\int_0^{2\pi} \int_0^\pi |Y_0^0|^2 \sin\theta d\theta d\phi = \int_0^{2\pi} \int_0^\pi \frac{1}{4\pi} \sin\theta d\theta d\phi \quad (9)$$

$$= 1 \quad (10)$$

$$\int_0^{2\pi} \int_0^\pi |Y_2^1|^2 \sin\theta d\theta d\phi = \frac{15}{8\pi} \int_0^{2\pi} \int_0^\pi \sin^3\theta \cos^2\theta d\theta d\phi \quad (11)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi \quad (12)$$

$$= 1 \quad (13)$$

The orthogonality can be checked in a similar way, by direct integration

$$\int_0^{2\pi} \int_0^\pi (Y_0^0)^* Y_2^1 \sin\theta d\theta d\phi = \sqrt{\frac{15}{32\pi^2}} \int_0^{2\pi} \int_0^\pi e^{i\phi} \sin^2\theta \cos\theta d\theta d\phi \quad (14)$$

$$= 0 \quad (15)$$

where the zero results from the first integral over θ .

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