SPHERICAL HARMONICS - NORMALIZATION

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We've already worked out the normalization of the spherical harmonics by finding the integral of the product of two associated Legendre functions. Here we find the normalization constant by a different route.

We know the general form of a spherical harmonic is

$$Y_l^m = B_l^m e^{im\phi} P_l^m(\cos\theta) \tag{1}$$

The raising and lowering operators in spherical coordinates are

$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right) \tag{2}$$

The effect of these operators on the spherical harmonics is:

$$L_\pm Y_l^m = \hbar\sqrt{(l\mp m)(l\pm m+1)}Y_l^{m\pm 1} \tag{3}$$

A formula for the derivative of the associated Legendre function is given in Griffiths's book:

$$\left(1-x^2\right)\frac{dP_l^m}{dx} = \sqrt{1-x^2}P_l^{m+1} - mxP_l^m \tag{4}$$

Using this with the chain rule, we get:

$$\frac{dP_l^m(\cos\theta)}{d\theta} = -P_l^{m+1}(\cos\theta) + m\cot\theta P_l^m(\cos\theta) \tag{5}$$

Using 5 when 2 is applied to 1, we get for the action of the raising operator

$$L_+ Y_l^m = \hbar e^{i(m+1)\phi} B_l^m\left(-P_l^{m+1} + m\cot\theta P_l^m - m\cot\theta P_l^m\right) \tag{6}$$

$$= -\hbar e^{i(m+1)\phi} B_l^m P_l^{m+1} \tag{7}$$

Equating this with 3 we get the recursion relation

$$B_l^{m+1}\sqrt{(l-m)(l+m+1)}P_l^{m+1} = -B_l^m P_l^{m+1} \tag{8}$$

We get

$$B_l^{m+1} = -\frac{B_l^m}{\sqrt{(l-m)(l+m+1)}} \tag{9}$$

If we start at $m=0$ and define $B_l^0 \equiv c(l)$, then $B_l^1 = -c(l)/\sqrt{l(l+1)}$, $B_l^2 = c(l)/\sqrt{l(l-1)(l+1)(l+2)}$ and in general for positive m:

$$B_l^m = (-1)^m \frac{c(l)}{\sqrt{l(l-1)\ldots(l-m+1)\times(l+1)(l+2)\ldots(l+m)}} \tag{10}$$

$$= (-1)^m c(l)\sqrt{\frac{(l-m)!}{(l+m)!}} \tag{11}$$

For negative m we use the lowering operator L_-. The convention for P_l^m used in Griffiths is that $P_l^{-m} = P_l^m$. When m is negative, we can write 5 as

$$\frac{dP_l^{|m|}(\cos\theta)}{d\theta} = -P_l^{-|m|+1}(\cos\theta) - |m|\cot\theta P_l^{|m|}(\cos\theta) \tag{12}$$

$$= -P_l^{|m|-1}(\cos\theta) - |m|\cot\theta P_l^{|m|}(\cos\theta) \tag{13}$$

That is, if we take m to be *positive*, then

$$\frac{dP_l^m(\cos\theta)}{d\theta} = -P_l^{m-1}(\cos\theta) - m\cot\theta P_l^m(\cos\theta) \tag{14}$$

Applying the lowering operators above and using this formula, we get the following recursion formula:

$$B_l^{m-1}\sqrt{(l+m)(l-m+1)}P_l^{m-1} = -B_l^m P_l^{m-1} \tag{15}$$

We get

$$B_l^{m-1} = -\frac{B_l^m}{\sqrt{(l+m)(l-m+1)}} \tag{16}$$

Again, starting with $m=0$ and going downwards we get for negative m:

$$B_l^m = (-1)^m \frac{c(l)}{\sqrt{l(l-1)\ldots(l-|m|+1)\times(l+1)(l+2)\ldots(l+|m|)}} \tag{17}$$

$$= (-1)^m c(l)\sqrt{\frac{(l-|m|)!}{(l+|m|)!}} \tag{18}$$

To find $c(l)$, we note from an earlier example that

$$P_l^l(\cos\theta) = \frac{\sin^l\theta}{2^l l!}(2l)! \tag{19}$$

and from above we know that

$$Y_l^l = B_l^l e^{il\phi} P_l^l(\cos\theta) \tag{20}$$

$$= (-1)^l c(l) \frac{1}{\sqrt{(2l)!}} e^{il\phi} \frac{\sin^l\theta}{2^l l!}(2l)! \tag{21}$$

$$= \sqrt{\frac{(2l+1)!}{4\pi}} \frac{1}{2^l l!} e^{il\phi} \sin^l\theta \tag{22}$$

where the third line comes from another earlier calculation where we worked out the spherical harmonic at the top of the ladder. Equating the last two right-hand-sides gives

$$c(l) = (-1)^l \sqrt{\frac{2l+1}{4\pi}} \tag{23}$$

so the final answer for B_l^m is

$$B_l^m = (-1)^{l+m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-|m|)!}{(l+|m|)!}} \tag{24}$$

which agrees with our earlier result.

The pesky factor of $(-1)^{l+m}$ can be omitted or modified according to the convention since it makes no difference to any physical calculation involving the spherical harmonics (since it is only products of two harmonics that have physical meaning).