SPIN - EXPECTATION VALUES OF COMPONENTS

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The most general spin state for a spin $\frac{1}{2}$ particle is

$$\chi=\left[\begin{array}{c} a\\ b \end{array}\right] \tag{1}$$

From this we can calculate the expectation values of the spin components. By direct matrix multiplication using the spin matrices, we get

$$\langle S_x\rangle=\frac{\hbar}{2}\left(\begin{array}{cc} a^* & b^*\end{array}\right)\left(\begin{array}{cc} 0 & 1\\ 1 & 0 \end{array}\right)\left(\begin{array}{c} a\\ b \end{array}\right)=\frac{\hbar}{2}(a^*b+b^*a) \tag{2}$$

This is a real quantity since it is the sum of a number and its complex conjugate. Similarly

$$\langle S_y\rangle=\frac{\hbar}{2}\left(\begin{array}{cc} a^* & b^*\end{array}\right)\left(\begin{array}{cc} 0 & -i\\ i & 0 \end{array}\right)\left(\begin{array}{c} a\\ b \end{array}\right)=i\frac{\hbar}{2}(b^*a-a^*b) \tag{3}$$

This is also real, since the difference of a number and its complex conjugate is pure imaginary, so multiplying by i gives a real number.

Finally,

$$\langle S_z\rangle=\frac{\hbar}{2}\left(\begin{array}{cc} a^* & b^*\end{array}\right)\left(\begin{array}{cc} 1 & 0\\ 0 & -1 \end{array}\right)\left(\begin{array}{c} a\\ b \end{array}\right)=\frac{\hbar}{2}(|a|^2-|b|^2) \tag{4}$$

Since $S_x^2=S_y^2=S_z^2=\frac{\hbar^2}{4}\left(\begin{array}{cc} 1 & 0\\ 0 & 1 \end{array}\right)$, $\langle S_x^2\rangle=\langle S_y^2\rangle=\langle S_z^2\rangle=|\chi|^2\hbar^2/4=\hbar^2/4$, and $\langle S_x^2\rangle+\langle S_y^2\rangle+\langle S_z^2\rangle=3\hbar^2/4=\langle S^2\rangle$ which agrees with our original result.