SPIN - STATISTICAL CALCULATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 4 September 2021.

We saw in the last post that spin 1/2 is represented by spin operators that are 2×2 matrices. Here we present an example of how some of the standard statistical calculations are done using these matrices.

Suppose we have an electron in the spin state

$$\chi = A\left[\begin{array}{c} 3i \\ 4 \end{array}\right] \tag{1}$$

This means that the electron's state is the sum of the spin up state multiplied by $3iA$ and the spin down state multiplied by $4A$. We can normalize the state by requiring its magnitude to be 1. That is

$$\chi^{\dagger}\chi = A^{*}A\left[\begin{array}{cc} -3i & 4 \end{array}\right]\left[\begin{array}{c} 3i \\ 4 \end{array}\right] = 25\left|A\right|^{2} = 1 \tag{2}$$

So $A = 1/5$.

We can find the mean values of the spin components by using an analogue of the method for calculating mean values with the spatial wave function. For an operator Q, instead of calculating $\int\boldsymbol{\Psi}^{*}Q\boldsymbol{\Psi}d^{3}\mathbf{r}$, we calculate $\chi^{\dagger}Q\chi$. Using the spin matrices from the last post, we get

$$\left\langle S_{x}\right\rangle = \frac{\hbar}{50}\left(\begin{array}{cc} -3i & 4 \end{array}\right)\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)\left(\begin{array}{c} 3i \\ 4 \end{array}\right) = 0 \tag{3}$$

$$\left\langle S_{y}\right\rangle = \frac{\hbar}{50}\left(\begin{array}{cc} -3i & 4 \end{array}\right)\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right)\left(\begin{array}{c} 3i \\ 4 \end{array}\right) = -\frac{24\hbar}{50} \tag{4}$$

$$\left\langle S_{z}\right\rangle = \frac{\hbar}{50}\left(\begin{array}{cc} -3i & 4 \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)\left(\begin{array}{c} 3i \\ 4 \end{array}\right) = -\frac{7\hbar}{50}$$

We can also get the standard deviations of the spin components. Since $S_{x}^{2} = S_{y}^{2} = S_{z}^{2} = \frac{\hbar^{2}}{4}\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$, $\langle S_{x}^{2}\rangle = \langle S_{y}^{2}\rangle = \langle S_{z}^{2}\rangle = (\hbar^{2}/4)|\chi|^{2} = \hbar^{2}/4$. So

$$\sigma_{S_x} = (\langle S_x^2 \rangle - \langle S_x \rangle^2)^{1/2} \tag{5}$$
$$= \frac{\hbar}{2} = \frac{25\hbar}{50} \tag{6}$$
$$\sigma_{S_y} = (\langle S_y^2 \rangle - \langle S_y \rangle^2)^{1/2} \tag{7}$$
$$= \frac{7\hbar}{50} \tag{8}$$
$$\sigma_{S_z} = (\langle S_z^2 \rangle - \langle S_z \rangle^2)^{1/2} \tag{9}$$
$$= \frac{24\hbar}{50} \tag{10}$$

From the generalized uncertainty principle, we get

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \left\langle [\hat{A}, \hat{B}] \right\rangle \right)^2 \tag{11}$$

The commutators for spin are:

$$[S_x, S_y] = i\hbar S_z \tag{12}$$
$$[S_y, S_z] = i\hbar S_x \tag{13}$$
$$[S_z, S_x] = i\hbar S_y \tag{14}$$

Working out the products, we get

$$\sigma_{S_x}\sigma_{S_y} = \frac{175}{2500}\hbar^2 \tag{15}$$
$$= \frac{\hbar}{2}\frac{7\hbar}{50} \tag{16}$$
$$= \frac{\hbar}{2}|\langle S_z \rangle| \tag{17}$$
$$\sigma_{S_y}\sigma_{S_z} = \frac{168}{2500}\hbar^2 \tag{18}$$
$$> \frac{\hbar}{2}|\langle S_x \rangle| = 0 \tag{19}$$
$$\sigma_{S_z}\sigma_{S_x} = \frac{600}{2500}\hbar^2 \tag{20}$$
$$= \frac{\hbar}{2}|\langle S_y \rangle| \tag{21}$$

so all relations are satisfied.