SPIN ONE-HALF ALONG AN ARBITRARY DIRECTION

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We've seen what the spin 1/2 matrices look like along the 3 rectangular coordinate axes. From this, we can derive an expression for the spin component along an arbitrary direction $\hat{\mathbf{r}}$. The unit radius vector is

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{1}$$

We can get S_r by combining S_x, S_y and S_z according to the formula for the radius vector:

$$\mathsf{S}_r = \mathsf{S}\cdot\hat{\mathbf{r}} \tag{2}$$

$$= \sin\theta\cos\phi S_x + \sin\theta\sin\phi S_y + \cos\theta S_z \tag{3}$$

By using the forms for the matrices derived earlier, and $\cos\phi \pm i\sin\phi = e^{\pm i\phi}$ we get

$$\mathsf{S}_r = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{4}$$

The eigenvalues of this matrix are calculated in the usual way

$$\begin{vmatrix} \frac{\hbar}{2}\cos\theta - \lambda & \frac{\hbar}{2}\sin\theta e^{-i\phi} \\ \frac{\hbar}{2}\sin\theta e^{i\phi} & -\frac{\hbar}{2}\cos\theta - \lambda \end{vmatrix} = -\frac{\hbar^2}{4}\left[\cos^2\theta + \sin^2\theta\right] + \lambda^2 = 0 \tag{5}$$

We get $\lambda = \pm\hbar/2$ as before so all is well at this stage.

To get the eigenspinors, we must solve

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta \pm 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \pm 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{6}$$

We get the equations

$$(\cos\theta \pm 1)\alpha + \sin\theta e^{-i\phi}\beta = 0 \tag{7}$$

$$\sin\theta e^{i\phi}\alpha - (-\cos\theta \pm 1)\beta = 0 \tag{8}$$

The two solutions (one for each sign) are

$$\beta_+ = -e^{i\phi}\frac{\cos\theta - 1}{\sin\theta}\alpha_+ \tag{9}$$

$$\beta_- = -e^{i\phi}\frac{\cos\theta + 1}{\sin\theta}\alpha_- \tag{10}$$

We can use the double-angle trig identities to simplify these expressions:

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) \tag{11}$$

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) \tag{12}$$

Substituting these together with $\cos^2(\theta/2) + \sin^2(\theta/2) = 1$ and simplifying leads to

$$\beta_+ = e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}\alpha_+ \tag{13}$$

$$\beta_- = -e^{i\phi}\frac{\cos(\theta/2)}{\sin(\theta/2)}\alpha_- \tag{14}$$

The eigenspinors should be normalized, so

$$|\beta_+|^2 + |\alpha_+|^2 = |\alpha_+|^2\left(\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} + 1\right) \tag{15}$$

$$= |\alpha_+|^2\left(\frac{\sin^2(\theta/2) + \cos^2(\theta/2)}{\cos^2(\theta/2)}\right) \tag{16}$$

$$= \frac{|\alpha_+|^2}{\cos^2(\theta/2)} \tag{17}$$

$$= 1 \tag{18}$$

Thus we can take

$$\alpha_+ = \cos\frac{\theta}{2} \tag{19}$$

Other answers are possible, since we can multiply α_+ by any complex exponential, as all that is important is that its magnitude is 1.

A similar calculation for the other solution leads to

$$\frac{|\alpha_-|^2}{\sin^2(\theta/2)} = 1 \tag{20}$$

We can take

$$\alpha_- = \sin\frac{\theta}{2} \tag{21}$$

These choices lead to

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{22}$$

$$\chi_-^{(r)} = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix} \tag{23}$$

If we want the answer in Griffiths, we would choose $\alpha_- = e^{-i\phi}\sin\frac{\theta}{2}$, which leads to the answer:

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{24}$$

$$\chi_-^{(r)} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{25}$$

The phase difference between the two components is the same in each solution.

Shankar's equations 14.3.28 use a slightly different phase, giving

$$|\hat{n}+\rangle = \begin{bmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{26}$$

$$|\hat{n}-\rangle = \begin{bmatrix} -\sin\frac{\theta}{2}e^{-i\phi/2} \\ \cos\frac{\theta}{2}e^{i\phi/2} \end{bmatrix} \tag{27}$$

We can calculate $\langle\mathbf{S}\rangle$ by using the spin matrices

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 01 \\ 10 \end{bmatrix} \tag{28}$$

$$S_y = \frac{\hbar}{2}\begin{bmatrix} 0-i \\ i0 \end{bmatrix} \tag{29}$$

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 10 \\ 0-1 \end{bmatrix} \tag{30}$$

We have

$$\langle\hat{n}+|S_x|\hat{n}+\rangle = \frac{\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{cc}0 & 1\\ 1 & 0\end{array}\right]\left[\begin{array}{c}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{array}\right] \tag{31}$$

$$= \frac{\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{c}\sin\frac{\theta}{2}e^{i\phi/2}\\ \cos\frac{\theta}{2}e^{-i\phi/2}\end{array}\right] \tag{32}$$

$$= \frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(e^{i\phi}+e^{-i\phi}\right) \tag{33}$$

$$= \frac{\hbar}{2}\sin\theta\cos\phi \tag{34}$$

$$\langle\hat{n}+|S_y|\hat{n}+\rangle = \frac{\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{cc}0 & -i\\ i & 0\end{array}\right]\left[\begin{array}{c}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{array}\right] \tag{35}$$

$$= \frac{i\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{c}-\sin\frac{\theta}{2}e^{i\phi/2}\\ \cos\frac{\theta}{2}e^{-i\phi/2}\end{array}\right] \tag{36}$$

$$= -\frac{\hbar}{2i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(-e^{i\phi}+e^{-i\phi}\right) \tag{37}$$

$$= \frac{\hbar}{2}\sin\theta\sin\phi \tag{38}$$

$$\langle\hat{n}+|S_z|\hat{n}+\rangle = \frac{\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{cc}1 & 0\\ 0 & -1\end{array}\right]\left[\begin{array}{c}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{i\phi/2}\end{array}\right] \tag{39}$$

$$= \frac{\hbar}{2}\left[\begin{array}{cc}\cos\frac{\theta}{2}e^{i\phi/2} & \sin\frac{\theta}{2}e^{-i\phi/2}\end{array}\right]\left[\begin{array}{c}\cos\frac{\theta}{2}e^{-i\phi/2}\\ -\sin\frac{\theta}{2}e^{i\phi/2}\end{array}\right] \tag{40}$$

$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right) \tag{41}$$

$$= \frac{\hbar}{2}\cos\theta \tag{42}$$

Thus the expectation values of the various spin components are their projections onto the three coordinate axes, as we might expect.

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