SPIN ONE-HALF PARTICLE IN A MAGNETIC FIELD

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In classical electromagnetic theory, the motion of charge (as in the current in a wire) gives rise to a magnetic field. In particular, if we have a charged object that is spinning about some axis, the motion of the charge gives the object a magnetic dipole moment μ which is related to the spin angular momentum $\mathbf{S}$ by the *gyromagnetic ratio* γ:

$$\mu = \gamma \mathbf{S} \tag{1}$$

The value of γ depends on the charge, shape and mass of the object. The interaction energy between a magnetic dipole and a magnetic field $\mathbf{B}$ is:

$$H = -\mu \cdot \mathbf{B} \tag{2}$$

so for a spinning charged particle whose centre of mass is at rest, we get

$$H = -\gamma \mathbf{S} \cdot \mathbf{B} \tag{3}$$

All this is classical (non-quantum) physics, but as usual, we can translate the equations into quantum theory in a straightforward way. If we take the spin to be a matrix, and consider spin 1/2, then

$$\mathsf{H} = -\gamma \mathbf{B} \cdot \mathsf{S} \tag{4}$$

If we take the magnetic field to be a constant and aligned in the z direction so that $\mathbf{B} = B_0 \hat{\mathbf{k}}$, then

$$\mathsf{H} = -\gamma B_0 \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{5}$$

Since the energy is just a constant multiplied by S_z, the eigenspinors are the same as those of S_z and the corresponding energies are

$$E_{\mp} = \pm \gamma B_0 \frac{\hbar}{2} \tag{6}$$

where E_- corresponds to the energy is the spin down state and $E_+$to spin up.

We can write the time-dependent solution in the same way as for the spatial wave function: we multiply each stationary state by $e^{-iEt/\hbar}$ and add up all the terms to get the general solution. In the example here, the general state is

$$\chi(t) = a\chi_+ e^{i\gamma B_0 t/2} + b\chi_- e^{-i\gamma B_0 t/2} = \left[\begin{array}{c} ae^{i\gamma B_0 t/2} \\ be^{-i\gamma B_0 t/2} \end{array}\right] \tag{7}$$

The constants a and b are determined from the initial conditions and from normalization, so they have to satisfy $|a|^2 + |b|^2 = 1$. The most general solution is then

$$\chi(t) = \left[\begin{array}{c} Ae^{i\beta}e^{i\gamma B_0 t/2} \\ Be^{i\delta}e^{-i\gamma B_0 t/2} \end{array}\right] \tag{8}$$

where A, B, β and δ are real constants determined by the initial conditions. From the normalization condition, we must have $A^2 + B^2 = 1$, so we might as well define

$$A = \cos\frac{\alpha}{2} \tag{9}$$

$$B = \sin\frac{\alpha}{2} \tag{10}$$

(The reason for using half angles is that α is an angle that turns up in a discussion of Larmor precession, which we won't get into here.)

If we take $\beta = \delta = 0$, then we can consider the case

$$\chi(t) = \left[\begin{array}{c} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{array}\right] \tag{11}$$

From this, we can work out the probabilities of getting various values for each of the spin components. We need to express $\chi(t)$ in terms of the eigenspinors of S_x:

$$\left(\begin{array}{c} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{array}\right) = \frac{a}{\sqrt{2}}\left(\begin{array}{c} 1 \\ 1 \end{array}\right) + \frac{b}{\sqrt{2}}\left(\begin{array}{c} 1 \\ -1 \end{array}\right) \tag{12}$$

This gives us a set of simultaneous equations in a and b which can be solved to give:

$$a = \frac{\sqrt{2}}{2}\left(\cos(\alpha/2)e^{i\gamma B_0 t/2} + \sin(\alpha/2)e^{-i\gamma B_0 t/2}\right) \tag{13}$$

$$b = \frac{\sqrt{2}}{2}\left(\cos(\alpha/2)e^{i\gamma B_0 t/2} - \sin(\alpha/2)e^{-i\gamma B_0 t/2}\right) \tag{14}$$

The probability of finding S_x in state $\hbar/2$ is therefore $|a|^2$ which works out to

$$|a|^2 = \frac{1}{2}(\cos(\alpha/2)e^{-i\gamma B_0 t/2} + \sin(\alpha/2)e^{i\gamma B_0 t/2})(\cos(\alpha/2)e^{i\gamma B_0 t/2} + \sin(\alpha/2)e^{-i\gamma B_0 t/2}) \tag{15}$$

$$= \frac{1}{2}(1 + \sin(\alpha/2)\cos(\alpha/2)(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t})) \tag{16}$$

$$= \frac{1}{2}(1 + 2\sin(\alpha/2)\cos(\alpha/2)\cos(\gamma B_0 t)) \tag{17}$$

$$= \frac{1}{2}(1 + \sin\alpha\cos(\gamma B_0 t)) \tag{18}$$

For completeness, the probability of getting $-\hbar/2$ is

$$|b|^2 = \frac{1}{2}(1 - \sin\alpha\cos(\gamma B_0 t)) \tag{19}$$

For S_y we can use the eigenspinors to get

$$\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{a}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \tag{20}$$

$$a = \frac{\sqrt{2}}{2}\left(\cos(\alpha/2)e^{i\gamma B_0 t/2} - i\sin(\alpha/2)e^{-i\gamma B_0 t/2}\right) \tag{21}$$

$$b = \frac{\sqrt{2}}{2}\left(\cos(\alpha/2)e^{i\gamma B_0 t/2} + i\sin(\alpha/2)e^{-i\gamma B_0 t/2}\right) \tag{22}$$

The probability of finding S_y in state $\hbar/2$ is therefore $|a|^2$ which works out to

$$|a|^2 = \frac{1}{2}(1 - \sin\alpha\sin(\gamma B_0 t)) \tag{23}$$

For S_z we have

$$\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{24}$$

so the probability of measuring S_z as $\hbar/2$ is

$$|a|^2 = \cos^2(\alpha/2) \tag{25}$$