

## SPIN ONE-HALF PARTICLE IN A MAGNETIC FIELD

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In classical electromagnetic theory, the motion of charge (as in the current in a wire) gives rise to a magnetic field. In particular, if we have a charged object that is spinning about some axis, the motion of the charge gives the object a magnetic dipole moment  $\mu$  which is related to the spin angular momentum  $\mathbf{S}$  by the *gyromagnetic ratio*  $\gamma$ :

$$\mu = \gamma \mathbf{S} \quad (1)$$

The value of  $\gamma$  depends on the charge, shape and mass of the object. The interaction energy between a magnetic dipole and a magnetic field  $\mathbf{B}$  is:

$$H = -\mu \cdot \mathbf{B} \quad (2)$$

so for a spinning charged particle whose centre of mass is at rest, we get

$$H = -\gamma \mathbf{S} \cdot \mathbf{B} \quad (3)$$

All this is classical (non-quantum) physics, but as usual, we can translate the equations into quantum theory in a straightforward way. If we take the spin to be a matrix, and consider spin 1/2, then

$$H = -\gamma \mathbf{B} \cdot \mathbf{S} \quad (4)$$

If we take the magnetic field to be a constant and aligned in the  $z$  direction so that  $\mathbf{B} = B_0 \hat{\mathbf{k}}$ , then

$$H = -\gamma B_0 \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (5)$$

Since the energy is just a constant multiplied by  $S_z$ , the eigenspinors are the same as those of  $S_z$  and the corresponding energies are

$$E_{\mp} = \pm \gamma B_0 \frac{\hbar}{2} \quad (6)$$

where  $E_-$  corresponds to the energy is the spin down state and  $E_+$  to spin up.

We can write the time-dependent solution in the same way as for the spatial wave function: we multiply each stationary state by  $e^{-iEt/\hbar}$  and add up all the terms to get the general solution. In the example here, the general state is

$$\chi(t) = a\chi_+ e^{i\gamma B_0 t/2} + b\chi_- e^{-i\gamma B_0 t/2} = \begin{bmatrix} ae^{i\gamma B_0 t/2} \\ be^{-i\gamma B_0 t/2} \end{bmatrix} \quad (7)$$

The constants  $a$  and  $b$  are determined from the initial conditions and from normalization, so they have to satisfy  $|a|^2 + |b|^2 = 1$ . The most general solution is then

$$\chi(t) = \begin{bmatrix} Ae^{i\beta} e^{i\gamma B_0 t/2} \\ Be^{i\delta} e^{-i\gamma B_0 t/2} \end{bmatrix} \quad (8)$$

where  $A$ ,  $B$ ,  $\beta$  and  $\delta$  are real constants determined by the initial conditions. From the normalization condition, we must have  $A^2 + B^2 = 1$ , so we might as well define

$$A = \cos \frac{\alpha}{2} \quad (9)$$

$$B = \sin \frac{\alpha}{2} \quad (10)$$

(The reason for using half angles is that  $\alpha$  is an angle that turns up in a discussion of Larmor precession, which we won't get into here.)

If we take  $\beta = \delta = 0$ , then we can consider the case

$$\chi(t) = \begin{bmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{bmatrix} \quad (11)$$

From this, we can work out the probabilities of getting various values for each of the spin components. We need to express  $\chi(t)$  in terms of the eigenspinors of  $S_x$ :

$$\begin{pmatrix} \cos(\alpha/2) e^{i\gamma B_0 t/2} \\ \sin(\alpha/2) e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (12)$$

This gives us a set of simultaneous equations in  $a$  and  $b$  which can be solved to give:

$$a = \frac{\sqrt{2}}{2} \left( \cos(\alpha/2) e^{i\gamma B_0 t/2} + \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right) \quad (13)$$

$$b = \frac{\sqrt{2}}{2} \left( \cos(\alpha/2) e^{i\gamma B_0 t/2} - \sin(\alpha/2) e^{-i\gamma B_0 t/2} \right) \quad (14)$$

The probability of finding  $S_x$  in state  $\hbar/2$  is therefore  $|a|^2$  which works out to

$$|a|^2 = \frac{1}{2}(\cos(\alpha/2)e^{-i\gamma B_0 t/2} + \sin(\alpha/2)e^{i\gamma B_0 t/2})(\cos(\alpha/2)e^{i\gamma B_0 t/2} + \sin(\alpha/2)e^{-i\gamma B_0 t/2}) \quad (15)$$

$$= \frac{1}{2}(1 + \sin(\alpha/2)\cos(\alpha/2)(e^{i\gamma B_0 t} + e^{-i\gamma B_0 t})) \quad (16)$$

$$= \frac{1}{2}(1 + 2\sin(\alpha/2)\cos(\alpha/2)\cos(\gamma B_0 t)) \quad (17)$$

$$= \frac{1}{2}(1 + \sin\alpha\cos(\gamma B_0 t)) \quad (18)$$

For completeness, the probability of getting  $-\hbar/2$  is

$$|b|^2 = \frac{1}{2}(1 - \sin\alpha\cos(\gamma B_0 t)) \quad (19)$$

For  $S_y$  we can use the eigenspinors to get

$$\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} = \frac{a}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{b}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad (20)$$

$$a = \frac{\sqrt{2}}{2} \left( \cos(\alpha/2)e^{i\gamma B_0 t/2} - i\sin(\alpha/2)e^{-i\gamma B_0 t/2} \right) \quad (21)$$

$$b = \frac{\sqrt{2}}{2} \left( \cos(\alpha/2)e^{i\gamma B_0 t/2} + i\sin(\alpha/2)e^{-i\gamma B_0 t/2} \right) \quad (22)$$

The probability of finding  $S_y$  in state  $\hbar/2$  is therefore  $|a|^2$  which works out to

$$|a|^2 = \frac{1}{2}(1 - \sin\alpha\sin(\gamma B_0 t)) \quad (23)$$

For  $S_z$  we have

$$\begin{pmatrix} \cos(\alpha/2)e^{i\gamma B_0 t/2} \\ \sin(\alpha/2)e^{-i\gamma B_0 t/2} \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (24)$$

so the probability of measuring  $S_z$  as  $\hbar/2$  is

$$|a|^2 = \cos^2(\alpha/2) \quad (25)$$

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