TIME-DEPENDENT PERTURBATION THEORY FOR A MULTI-LEVEL SYSTEM

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So far, we've looked at time-dependent perturbation theory only for a two-state system. We can generalize this to a multi-state system fairly easily, by following the derivation in the two-state case. We start with the exact system:

$$H_0\psi_n = E_n\psi_n \tag{1}$$

where H_0 is the time-independent hamiltonian and the states (there are now assumed to be an arbitrary number of states) are orthonormal:

$$\langle\psi_n|\psi_m\rangle = \delta_{nm} \tag{2}$$

At time $t=0$ we introduce a time-dependent perturbation $H'(t)$ so we now have

$$H(t) = H_0 + H'(t) \tag{3}$$

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{4}$$

Since the eigenstates ψ_n form a complete set, we can write the solution as a linear combination, but with time-dependent coefficients $c_n(t)$:

$$\Psi(t) = \sum_n c_n(t)\psi_n e^{-iE_nt/\hbar} \tag{5}$$

Plugging this into 4 and using 1 we get

$$\sum_n c_n(t) H\psi_n e^{-iE_nt/\hbar} = \sum_n c_n(t)\left(H_0 + H'\right)\psi_n e^{-iE_nt/\hbar} \tag{6}$$

$$= \sum_n c_n(t)\left(E_n + H'\right)\psi_n e^{-iE_nt/\hbar} \tag{7}$$

$$= \sum_n \left(E_n c_n(t) + i\hbar\dot{c}_n(t)\right)\psi_n e^{-iE_nt/\hbar} \tag{8}$$

where the last line is the time derivative of 5. Cancelling off the first term from each side, we get

$$\sum_n c_n(t) H' \psi_n e^{-iE_n t/\hbar} = i\hbar \sum_n \dot{c}_n(t) \psi_n e^{-iE_n t/\hbar} \tag{9}$$

We can now use the usual trick of multiplying both sides by ψ_m and integrating over all space (remember that the c_ns are independent of space), using 2:

$$i\hbar \dot{c}_m(t) e^{-iE_m t/\hbar} = \sum_n c_n(t) \left\langle \psi_m \left| H' \right| \psi_n \right\rangle e^{-iE_n t/\hbar} \tag{10}$$

$$\dot{c}_m = -\frac{i}{\hbar} \sum_n c_n(t) \left\langle \psi_m \left| H' \right| \psi_n \right\rangle e^{i(E_m - E_n)t/\hbar} \tag{11}$$

$$= -\frac{i}{\hbar} \sum_n c_n(t) H'_{mn} e^{i(E_m - E_n)t/\hbar} \tag{12}$$

That's the set of ODEs that would need to be solved to get an exact solution. At this point, we can introduce perturbation theory in the same way as in the two-state case. We assume that the system starts out in state ψ_N so that $c_n(0) = \delta_{nN}$. Using the iterative solution, the zeroth order coefficients are

$$c_n^{(0)} = \delta_{nN} \tag{13}$$

We can plug these into 12 to get the first order terms:

$$\dot{c}_N^{(1)} = -\frac{i}{\hbar} H'_{NN} \tag{14}$$

$$c_N^{(1)} = 1 - \frac{i}{\hbar} \int_0^t H'_{NN}(t') \, dt' \tag{15}$$

where the 1 is the constant of integration chosen to match the initial condition. For the other coefficients we have, for $m \neq N$:

$$c_m(t) = -\frac{i}{\hbar} \int_0^t H'_{mN} e^{i(E_m - E_N)t'/\hbar} dt' \tag{16}$$

where again the constant of integration is chosen for the initial condition 13.

Now suppose that the perturbation H' is constant in the interval $[0, t]$ (that is, it's switched on at $t = 0$ and off at a later time t). Then the matrix elements H_{mN} are constant over the domain of the integrals, so we get, for $m \neq N$:

$$c_m(t) = -\frac{i}{\hbar} H'_{mN} \int_0^t e^{i(E_m - E_N)t'/\hbar} dt' \tag{17}$$

$$= -\frac{i}{\hbar} H'_{mN} \frac{\hbar}{i(E_m - E_N)} \left(e^{i(E_m - E_N)t/\hbar} - 1 \right) \tag{18}$$

$$= -\frac{2iH'_{mN}}{(E_m - E_N)} e^{i(E_m - E_N)t/2\hbar} \sin\frac{(E_m - E_N)t}{2\hbar} \tag{19}$$

The first order probability of a transition from the initial state ψ_N to a different state ψ_m is therefore

$$|c_m(t)|^2 = \frac{4\left|H'_{mN}\right|^2}{(E_m - E_N)^2} \sin^2\frac{(E_m - E_N)t}{2\hbar} \tag{20}$$

As a different example, suppose the perturbation is sinusoidal in time, so that

$$H'(t) = V\cos(\omega t) \tag{21}$$

where V is some function of space only. Then we get

$$c_m(t) = -\frac{i}{\hbar} V_{mN} \int_0^t \cos\left(\omega t'\right) e^{i(E_m - E_N)t'/\hbar} dt' \tag{22}$$

$$= -\frac{iV_{mN}}{2\hbar} \int_0^t \left(e^{i\omega t'} + e^{-i\omega t'} \right) e^{i(E_m - E_N)t'/\hbar} dt' \tag{23}$$

$$= -\frac{V_{mN}}{2} \left[\frac{e^{i(\hbar\omega + E_m - E_N)t/\hbar} - 1}{\hbar\omega + E_m - E_N} + \frac{e^{i(-\hbar\omega + E_m - E_N)t/\hbar} - 1}{-\hbar\omega + E_m - E_N} \right] \tag{24}$$

If we make the approximation that c_m is sharply peaked around the values where one of the denominators is zero, then we can have a transition only when $E_m = E_N \pm \hbar\omega$. In that case, we have

$$c_m(t) = -\frac{V_{mN}}{2} \frac{e^{i(\pm\hbar\omega + E_m - E_N)t/\hbar} - 1}{\pm\hbar\omega + E_m - E_N} \tag{25}$$

$$= -V_{mN} i \frac{e^{i(E_m - E_N \pm \hbar\omega)t/2\hbar}}{E_m - E_N \pm \hbar\omega} \sin\frac{(E_m - E_N \pm \hbar\omega)t}{2\hbar} \tag{26}$$

$$= -V_{mN} i \frac{e^{i(E_m - E_N \pm \hbar\omega)t/2\hbar}}{E_N - E_m \pm \hbar\omega} \sin\frac{(E_N - E_m \pm \hbar\omega)t}{2\hbar} \tag{27}$$

where the last step comes from multiplying top and bottom by -1 and using $\sin(-x) = -\sin x$. The transition probability is then

$$|c_m(t)|^2 = \frac{|V_{mN}|^2}{(E_N - E_m \pm \hbar\omega)^2} \sin^2 \frac{(E_N - E_m \pm \hbar\omega)\, t}{2\hbar} \tag{28}$$

To get the rate for stimulated emission, the derivation is exactly the same as previously, with $V_{mN} = -\mathfrak{p}E_0$, where $E_0 = \hbar\omega_0$ is the energy of a photon of the stimulating radiation. Averaging over polarization and propagation directions again works out exactly the same as before (as in Griffiths section 9.2.3) so the result must be the same. The stimulated emission rate is

$$R = \frac{\pi |\mathfrak{p}|^2}{3\epsilon_0\hbar^2}\rho(\omega_0) \tag{29}$$