

## TRANSLATION INVARIANCE IN TWO DIMENSIONS

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In preparation for an examination of rotation invariance, we'll have a look at translational invariance in two dimensions. We can apply much of what we did with translation in one dimension, where we showed that the momentum  $P$  is the generator of translations. In particular, the translation operator  $T(\varepsilon)$  for an infinitesimal translation  $\varepsilon$  is

$$T(\varepsilon) = I - \frac{i\varepsilon}{\hbar}P \quad (1)$$

In two dimensions, we can write an infinitesimal translation as  $\boldsymbol{\delta a}$  where

$$\boldsymbol{\delta a} = \delta a_x \hat{\mathbf{x}} + \delta a_y \hat{\mathbf{y}} \quad (2)$$

In one dimension, we showed earlier that

$$\langle x | T(\varepsilon) | \psi \rangle = \psi(x - \varepsilon) \quad (3)$$

The analogous relation in two dimensions is

$$\langle x, y | T(\boldsymbol{\delta a}) | \psi \rangle = \psi(x - \delta a_x, y - \delta a_y) \quad (4)$$

We can verify that the correct form for  $T(\boldsymbol{\delta a})$  is

$$T(\boldsymbol{\delta a}) = I - \frac{i}{\hbar} \boldsymbol{\delta a} \cdot \mathbf{P} \quad (5)$$

$$= I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y) \quad (6)$$

Using the representation of momentum in the position basis, which is

$$P_x = -i\hbar \frac{\partial}{\partial x} \quad (7)$$

$$P_y = -i\hbar \frac{\partial}{\partial y} \quad (8)$$

the LHS of 4 is, using  $\langle x, y | \psi \rangle = \psi(x, y)$ :

$$\langle x, y | T(\boldsymbol{\delta a}) | \psi \rangle = \left\langle x, y \left| I - \frac{i}{\hbar} (\delta a_x P_x + \delta a_y P_y) \right| \psi \right\rangle \quad (9)$$

$$= \psi(x, y) - \delta a_x \frac{\partial \psi}{\partial x} - \delta a_y \frac{\partial \psi}{\partial y} \quad (10)$$

The last line is also what we get if we expand the RHS of 4 to first order in  $\boldsymbol{\delta a}$ , which verifies that 5 is correct, so that the two-dimensional momentum  $\mathbf{P}$  is the generator of two-dimensional translations.

We can apply the exponentiation technique we used in the one-dimensional case to obtain the translation operator for a finite translation in two dimensions. We need to be careful that we don't run into problems with non-commuting operators, but in view of 7 and 8 and the fact that derivatives with respect to different independent variables commute, we see that

$$[P_x, P_y] = 0 \quad (11)$$

We can divide a finite translation  $\mathbf{a}$  into  $N$  small steps, each of size  $\frac{\mathbf{a}}{N}$ , so that the translation is

$$T(\mathbf{a}) = \left( I - \frac{i}{\hbar N} \mathbf{a} \cdot \mathbf{P} \right)^N \quad (12)$$

Because the two components of momentum commute, we can take the limit of this expression to get the exponential form:

$$T(\mathbf{a}) = \lim_{N \rightarrow \infty} \left( I - \frac{i}{\hbar N} \mathbf{a} \cdot \mathbf{P} \right)^N = e^{-i\mathbf{a} \cdot \mathbf{P} / \hbar} \quad (13)$$

Again, because the two components of momentum commute, we can combine two translations, by  $\mathbf{a}$  and then by  $\mathbf{b}$ , to get

$$T(\mathbf{b})T(\mathbf{a}) = e^{-i\mathbf{b} \cdot \mathbf{P} / \hbar} e^{-i\mathbf{a} \cdot \mathbf{P} / \hbar} = e^{-i(\mathbf{a} + \mathbf{b}) \cdot \mathbf{P} / \hbar} = T(\mathbf{b} + \mathbf{a}) \quad (14)$$

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