

## TRANSLATION OPERATOR FROM PASSIVE TRANSFORMATIONS

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Post date: 19 July 2021.

We've seen that the translation operator  $T(\varepsilon)$  in quantum mechanics can be derived by considering the translation to be an active transformation, that is, a transformation where the state vectors, rather than the operators, get transformed according to

$$T(\varepsilon)|\psi\rangle = |\psi_\varepsilon\rangle \quad (1)$$

Using this approach, we found that

$$T(\varepsilon) = I - \frac{i\varepsilon}{\hbar}P \quad (2)$$

so that the momentum  $P$  is the generator of the transformation.

We can also derive  $T$  using a passive transformation, where the state vectors remain the same but the operators are transformed according to

$$T^\dagger(\varepsilon)XT(\varepsilon) = X + \varepsilon I \quad (3)$$

$$T^\dagger(\varepsilon)PT(\varepsilon) = P \quad (4)$$

This is equivalent to an active transformation since

$$\langle\psi|T^\dagger(\varepsilon)XT(\varepsilon)|\psi\rangle = \langle T(\varepsilon)\psi|X|T(\varepsilon)\psi\rangle \quad (5)$$

$$= \langle\psi_\varepsilon|X|\psi_\varepsilon\rangle \quad (6)$$

$$= x + \varepsilon \quad (7)$$

As before we start by taking

$$T(\varepsilon) = I - \frac{i\varepsilon}{\hbar}G \quad (8)$$

where  $G$  is some Hermitian operator, so that  $G^\dagger = G$ . Plugging this into 3 we get, keeping only terms up to order  $\varepsilon$ :

$$T^\dagger(\varepsilon)XT(\varepsilon) = \left(I + \frac{i\varepsilon}{\hbar}G\right)X\left(I - \frac{i\varepsilon}{\hbar}G\right) \quad (9)$$

$$= X + \frac{i\varepsilon}{\hbar}I(GX - XG) \quad (10)$$

$$= X - \frac{i\varepsilon}{\hbar}[X, G] \quad (11)$$

$$= X + \varepsilon I \quad (12)$$

where the last line is a requirement arising from 7.

Therefore

$$-\frac{i\varepsilon}{\hbar}[X, G] = \varepsilon I \quad (13)$$

$$[X, G] = i\hbar I \quad (14)$$

Since  $[X, P] = i\hbar$  we see that

$$G = P + f(X) \quad (15)$$

The extra  $f(X)$  is there because any function of  $X$  alone commutes with  $X$ , so

$$[X, G] = [X, P] + [X, f(X)] = i\hbar I + 0 \quad (16)$$

We can eliminate  $f(X)$  by considering 4.

$$T^\dagger(\varepsilon)PT(\varepsilon) = \left(I + \frac{i\varepsilon}{\hbar}G\right)P\left(I - \frac{i\varepsilon}{\hbar}G\right) \quad (17)$$

$$= P + \frac{i\varepsilon}{\hbar}I(GP - PG) \quad (18)$$

$$= P - \frac{i\varepsilon}{\hbar}[P, G] \quad (19)$$

$$= P \quad (20)$$

Thus we must have  $[P, G] = 0$ , which means that  $G$  must be a function of  $P$  alone. This means that the most general form for  $f(X)$  is  $f(X) = \text{constant}$ , but there's nothing to be gained by adding some non-zero constant to  $G$ , so we can take  $f(X) = 0$ . Thus we end up with the same form 2 that we got from the active transformation.

Translational invariance is the condition that the Hamiltonian is unaltered by a translation. In the passive representation this is stated by the condition

$$T^\dagger(\varepsilon)HT(\varepsilon) = H \quad (21)$$

Since translation is unitary, we can apply a theorem that is valid for any operator  $\Omega$  which can be expanded in powers of  $X$  and  $P$ . For any unitary operator  $U$ , we have

$$U^\dagger \Omega(X, P) U = \Omega(U^\dagger X U, U^\dagger P U) \quad (22)$$

This follows because for a unitary operator  $U^\dagger U = U U^\dagger = I$  so we can insert the product  $U U^\dagger$  anywhere we like. In particular, we can insert it between each pair of factors in every term of the power series expansion of  $\Omega$ , for example

$$U^\dagger X^2 P^2 U = U^\dagger X X P P U \quad (23)$$

$$= U^\dagger X U U^\dagger X U U^\dagger P U U^\dagger P U \quad (24)$$

$$= (U^\dagger X U)^2 (U^\dagger P U)^2 \quad (25)$$

For 21 this means that

$$T^\dagger(\varepsilon) H(X, P) T(\varepsilon) = H(X + \varepsilon I, P) = H(X, P) \quad (26)$$

As before, this leads to the condition

$$[P, H] = 0 \quad (27)$$

which means that  $P$  is conserved, according to Ehrenfest's theorem.

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