

TRANSLATIONAL INVARIANCE AND CONSERVATION OF MOMENTUM

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One consequence of the invariance of the Hamiltonian under translation is that the momentum and Hamiltonian commute:

$$[P, H] = 0 \quad (1)$$

In quantum mechanics, commuting quantities are simultaneously observable, and we can find a basis for the Hilbert space consisting of eigenstates of both P and H . We've seen that Ehrenfest's theorem allows us to conclude that for such a system, the average momentum is conserved so that $\langle \dot{P} \rangle = 0$. We can go a step further and state that if a system starts out in an eigenstate of P , then it remains in that eigenstate for all time.

First, we need to make a rather subtle observation, which is that

$$[P, H] = 0 \rightarrow [P, U(t)] = 0 \quad (2)$$

That is, if P and H commute, then P also commutes with the propagator $U(t)$. For a time-independent Hamiltonian, the propagator is

$$U(t) = e^{-iHt/\hbar} \quad (3)$$

Since this can be expanded in a power series in the Hamiltonian, condition 2 follows easily enough. What if the Hamiltonian is time-dependent? In this case, the propagator comes out to a time-ordered integral

$$U(t) = T \left\{ \exp \left[-\frac{i}{\hbar} \int_0^t H(t') dt' \right] \right\} \equiv \lim_{N \rightarrow \infty} \prod_{n=0}^{N-1} e^{-i\Delta H(n\Delta)/\hbar} \quad (4)$$

Here the time interval $[0, t]$ is divided into N time slices, each of length $\Delta = t/N$. As explained in the earlier post, the reason we can't just integrate the RHS directly by summing the exponents is that such a procedure works only if the operators in the exponents all commute with each other. If H is time-dependent, its forms at different times may not commute, so we can't get a simple closed form for $U(t)$.

However, if $[P, H(t)] = 0$ for all times, then P commutes with all the exponents on the RHS of 4, so we still get $[P, U(t)] = 0$. Another way of looking at this is by imposing the condition $[P, H(t)] = 0$ we're saying that if $H(t)$ can be expanded in a power series in X and P , it depends only on P , and not on X . This follows from the fact that

$$[X^n, P] = i\hbar n X^{n-1} \tag{5}$$

so that P does not commute with any power of X .

Given that 2 is valid for all Hamiltonians, then if we start in an eigenstate $|p\rangle$ of P , then

$$P|p\rangle = p|p\rangle \tag{6}$$

$$PU(t)|p\rangle = U(t)P|p\rangle \tag{7}$$

$$= U(t)p|p\rangle \tag{8}$$

$$= pU(t)|p\rangle \tag{9}$$

Thus $U(t)|p\rangle$ remains an eigenstate of P with the same eigenvalue p for all time. For a single particle moving in one dimension, the state $|p\rangle$ describes a free particle with momentum p (and thus a completely undetermined position).

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