TWO-DIMENSIONAL HARMONIC OSCILLATOR - COMPARISON WITH RECTANGULAR COORDINATES

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Post date: 23 July 2021.

In this post, we'll continue with the solution of the 2-d isotropic harmonic oscillator. In the previous post, we found that the radial equation R can be written as

$$R(y)=y^{|m|}e^{-y^{2}/2}U(y) \tag{1}$$

where the dimensionless variables are given by

$$y\equiv\sqrt{\frac{\mu\omega}{\hbar}}\rho \tag{2}$$

$$\varepsilon\equiv\frac{E}{\hbar\omega} \tag{3}$$

and U has a solution as a power series

$$U(y)=\sum_{r=0}^{\infty}C_{r}y^{r} \tag{4}$$

The coefficients C_r satisfy the recursion relation

$$C_{r+2}=\frac{2(r+|m|+1-\varepsilon)}{(r+2)^{2}}C_{r} \tag{5}$$

Only C_r for even r are non-zero.

In order for U to remain finite for large y, the series must terminate, which gives the allowable values for the energy as

$$E=\hbar\omega(n+1) \tag{6}$$

with

$$n\equiv 2k+|m| \tag{7}$$

and $k=0,1,2,\ldots$.

We can now compare the solution obtained in polar coordinates with our earlier solution in terms of rectangular coordinates. First, what are the possible values for m for a given energy $E = \hbar\omega(n+1)$? From the relation 7, we can look at even and odd n separately. For even n, k can take values $0, 1, \ldots, \frac{n}{2}-1, \frac{n}{2}$. The first $\frac{n}{2}$ of these values for k (that is, for $k = 0, 1, \ldots, \frac{n}{2}-1$) each allow two values of m such that $|m| = n-2k$, namely $m = \pm(n-2k)$. If $k = \frac{n}{2}$, then we must have $m = 0$. Thus for even n the total number of combinations is $2 \times \frac{n}{2} + 1 = n+1$.

For odd n, k can take on values $0, 1, \ldots, \frac{n-1}{2}$, giving a total of $\frac{n+1}{2}$ possible values. (If this isn't obvious, write it out for the first few values of odd n to see the pattern.) For each of these values of k, m can take on the two values $m = \pm(n-2k)$, thus there are again $2 \times \frac{n+1}{2} = n+1$ different combinations. Thus a state with energy $E = \hbar\omega(n+1)$ has a degeneracy $n+1$.

We can construct the actual eigenfunctions for a couple of values of n by plugging in the appropriate formulas. For $n = 0$ there is only one function, which we find by setting $k = m = 0$. From 4, we have

$$U_0(y) = C_0 \tag{8}$$

and from 1 we have

$$R_0(y) = C_0 e^{-y^2/2} \tag{9}$$

or, in terms of the original variables

$$R_0(\rho) = C_0 e^{-\mu\omega\rho^2/2\hbar} \tag{10}$$

The complete solution is given by

$$\begin{aligned}\psi_m(\rho,\phi) &= R(\rho)\,\Phi_m(\phi) &(11)\\ &= \frac{1}{\sqrt{2\pi}} R(\rho)\, e^{im\phi} &(12)\end{aligned}$$

so for $m = 0$ we have

$$\psi_0(\rho,\phi) = \frac{C_0}{\sqrt{2\pi}} e^{-\mu\omega\rho^2/2\hbar} \tag{13}$$

The constant C_0 can be found by normalizing:

$$1 = \int_0^\infty \int_0^{2\pi} |\psi_0|^2 \rho d\phi d\rho \tag{14}$$
$$= |C_0|^2 \int_0^\infty e^{-\mu\omega\rho^2/\hbar} \rho d\rho \tag{15}$$
$$= |C_0|^2 \frac{\hbar}{2\mu\omega} \tag{16}$$
$$C_0 = \sqrt{\frac{2\mu\omega}{\hbar}} \tag{17}$$
$$\psi_0(\rho,\phi) = \sqrt{\frac{\mu\omega}{\pi\hbar}} e^{-\mu\omega\rho^2/2\hbar} \tag{18}$$

This agrees with the earlier result in rectangular coordinates (eqn 26 in this post). This must be the case, since the $n = 0$ state is non-degenerate.

For $n = 1$, we have $k = 0$ and $m = \pm 1$ so we have two solutions:

$$\psi_1 = \frac{C_0}{\sqrt{2\pi}} \sqrt{\frac{\mu\omega}{\hbar}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{i\phi} \tag{19}$$
$$\psi_{-1} = \frac{C_0}{\sqrt{2\pi}} \sqrt{\frac{\mu\omega}{\hbar}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{-i\phi} \tag{20}$$

Again, we normalize

$$1 = \int_0^\infty \int_0^{2\pi} |\psi_{\pm 1}|^2 \rho d\phi d\rho \tag{21}$$
$$= \frac{\mu\omega}{\hbar} |C_0|^2 \int_0^\infty e^{-\mu\omega\rho^2/\hbar} \rho^3 d\rho \tag{22}$$
$$C_0 = \sqrt{\frac{2\mu\omega}{\hbar}} \tag{23}$$
$$\psi_{\pm 1} = \frac{\mu\omega}{\hbar\sqrt{\pi}} \rho e^{-\mu\omega\rho^2/2\hbar} e^{\pm i\phi} \tag{24}$$

These solutions are linear combinations of the corresponding solutions in rectangular coordinates:

$$\psi_{10} = \frac{\sqrt{2}\mu\omega}{\hbar\sqrt{\pi}} e^{-\mu\omega\rho^2/2\hbar} \rho \cos\phi \tag{25}$$
$$\psi_{01} = \frac{\sqrt{2}\mu\omega}{\hbar\sqrt{\pi}} e^{-\mu\omega\rho^2/2\hbar} \rho \sin\phi \tag{26}$$

The combinations are

$$\psi_{+1} = \frac{1}{\sqrt{2}}(\psi_{10} + i\psi_{01}) \tag{27}$$

$$\psi_{-1} = \frac{1}{\sqrt{2}}(\psi_{10} - i\psi_{01}) \tag{28}$$

The parity of the states is found from their behaviour under the transformation (in rectangular coordinates) $x \to -x$ and $y \to -y$. In polar coordinates this is equivalent to the transformation $\phi \to \phi + \pi$ and from 18 and 24 we see that

$$\psi_0(\rho, \phi + \pi) = \psi_0(\rho, \phi) \tag{29}$$

$$\psi_{\pm 1}(\rho, \phi + \pi) = \psi_{\pm 1}(\rho, \phi)\, e^{\pm \pi} \tag{30}$$

$$= -\psi_{\pm 1}(\rho, \phi) \tag{31}$$

Thus the parity of $n = 0$ is even, and that of $n = 1$ is odd. In general, since the ϕ dependence enters only through the term $e^{im\phi} = e^{in\phi}e^{-2ik\phi}$, we see that adding π to ϕ leaves the $e^{-2ik\phi}$ term unchanged and multiplies the $e^{in\phi}$ term by $e^{in\pi\phi} = (-1)^n$, so the parity of state n is $(-1)^n$.

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