TWO-DIMENSIONAL HARMONIC OSCILLATOR

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In this problem, we'll look at solving the 2-dimensional isotropic harmonic oscillator. The method of solution is similar to that used in the one-dimensional harmonic oscillator, so you may wish to refer back to that before proceeding.

The Hamiltonian is, in rectangular coordinates:

$$H=\frac{P_x^2+P_y^2}{2\mu}+\frac{1}{2}\mu\omega^2\left(X^2+Y^2\right) \tag{1}$$

The potential term is radially symmetric (it doesn't depend on the polar angle ϕ) so we have a problem of the form considered earlier. We saw there that for such potentials $[H,L_z]=0$. [If you don't believe this, you can grind through the calculations using the commutation relations for L_z with the rectangular momenta and coordinates, but I won't go through that here.]

As a result, L_z and H have simultaneous eigenfunctions of form

$$\psi(\rho,\phi)=R(\rho)\Phi_m(\phi) \tag{2}$$

where

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi} \tag{3}$$

The radial function satisfies the ODE

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}-\frac{m^2}{\rho^2}R\right)+V(\rho)R=ER \tag{4}$$

where in this case

$$V(\rho)=\frac{1}{2}\mu\omega^2\rho^2 \tag{5}$$

Thus the equation we must solve is

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{d\rho^2}+\frac{1}{\rho}\frac{d}{d\rho}-\frac{m^2}{\rho^2}\right)R+\frac{1}{2}\mu\omega^2\rho^2R=ER \tag{6}$$

To get a feel for the solution, we examine the behaviour in two limiting cases: $\rho \to 0$ and $\rho \to \infty$. We define the dimensionless variables

$$y \equiv \sqrt{\frac{\mu\omega}{\hbar}}\rho \tag{7}$$

$$\varepsilon \equiv \frac{E}{\hbar\omega} \tag{8}$$

This transforms 6 to

$$-\frac{\hbar^2}{2\mu}\left(\frac{\mu\omega}{\hbar}\right)\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}-\frac{m^2}{y^2}\right)R+\frac{1}{2}\hbar\omega y^2 R = ER \tag{9}$$

$$-\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}-\frac{m^2}{y^2}+2\varepsilon\right)R+y^2R = 0 \tag{10}$$

We can now look at $y \to 0$, and we neglect the terms $2\varepsilon R$ and $y^2 R$ to get

$$\left(\frac{d^2}{dy^2}+\frac{1}{y}\frac{d}{dy}-\frac{m^2}{y^2}\right)R = 0 \tag{11}$$

If we try a solution of form

$$R = y^{|m|} \tag{12}$$

we have

$$|m|\left(|m|-1\right)y^{|m|-2}+|m|\,y^{|m|-2}-m^2y^{|m|-2} = 0 \tag{13}$$

Thus 12 is indeed a solution in this limiting case.

For $y \to \infty$, we can ignore the terms $\frac{1}{y}\frac{d}{dy}$, $\frac{m^2}{y^2}R$ and $2\varepsilon R$ to get

$$-\frac{d^2}{dy^2}R+y^2R = 0 \tag{14}$$

or

$$R'' = y^2 R \tag{15}$$

We try a solution of form

$$R = y^a e^{-y^2/2} \tag{16}$$

where a is some constant. We find

$$R' = \left(ay^{a-1} - y^{a+1}\right)e^{-y^2/2} \tag{17}$$

$$R'' = \left(a(a-1)y^{a-2} - (a+1)y^a - ay^a + y^{a+2}\right)e^{-y^2/2} \tag{18}$$

$$= y^{a+2}\left(\frac{a(a-1)}{y^4} - \frac{2a+1}{y^2} + 1\right)e^{-y^2/2} \tag{19}$$

As $y \to \infty$, the last line tends to

$$R'' \to y^{a+2}e^{-y^2/2} = y^2 R \tag{20}$$

so in this limit 16 is a solution. We can therefore propose that R has the general form

$$R(y) = y^{|m|}e^{-y^2/2}U(y) \tag{21}$$

where U is a function to be determined by solving the exact ODE 10. We can get an ODE for U by substituting 21 into 10, although the calculation gets somewhat messy. As Shankar suggests, we can do this in two stages. First, we substitute

$$R = y^{|m|}f(y) \tag{22}$$

where

$$f(y) = e^{-y^2/2}U(y) \tag{23}$$

The required derivatives are (To make the notation simpler, I'll drop the absolute value signs around m; you should assume that wherever m occurs, it should really be $|m|$. We can replace the absolute value sign at the end.)

$$R' = my^{m-1}f + y^m f' \tag{24}$$

$$R'' = m(m-1)y^{m-2}f + 2my^{m-1}f' + y^m f'' \tag{25}$$

Plugging these into 10 we have

$$-\left(m(m-1)y^{m-2}f + 2my^{m-1}f' + y^m f''\right) - \left(my^{m-2}f + y^{m-1}f'\right) + \ldots \tag{26}$$

$$m^2 y^{m-2}f - 2\varepsilon y^m f + y^{m+2}f = 0 \tag{27}$$

Collecting terms and dividing through by $-y^m$, we get

$$f'' + f'\left(\frac{2m+1}{y}\right) + f\left(2\varepsilon - y^2\right) = 0 \tag{28}$$

We now get the derivatives of f:

$$f' = -ye^{-y^2/2}U + e^{-y^2/2}U' \tag{29}$$
$$= e^{-y^2/2}\left(U' - yU\right) \tag{30}$$
$$f'' = \left[-y\left(U' - yU\right) + U'' - U - yU'\right]e^{-y^2/2} \tag{31}$$
$$= \left(U'' - 2yU' + \left(y^2 - 1\right)U\right)e^{-y^2/2} \tag{32}$$

When we plug these into 28, the exponential factor cancels out, so we get

$$U'' - 2yU' + \left(y^2 - 1\right)U + \frac{2m+1}{y}\left(U' - yU\right) + U\left(2\varepsilon - y^2\right) = 0 \tag{33}$$

Collecting terms, we get, upon restoring the absolute values:

$$U'' + \left(\frac{2|m|+1}{y} - 2y\right)U' + (2\varepsilon - 2|m| - 2)U = 0 \tag{34}$$

We can solve this by using a power series of the form

$$U(y) = \sum_{r=0}^{\infty} C_r y^r \tag{35}$$

where the coefficients C_r are constants.

The derivatives are

$$U' = \sum_{r=0}^{\infty} C_r r y^{r-1} \tag{36}$$
$$= 0 + C_1 + 2C_2 y + 3C_3 + y^2 + \ldots \tag{37}$$
$$= \sum_{r=0}^{\infty} C_{r+1}(r+1)y^r \tag{38}$$
$$U'' = \sum_{r=0}^{\infty} C_{r+1} r(r+1)y^{r-1} \tag{39}$$
$$= 0 + (1)(2)C_2 + (2)(3)C_3 y + \ldots \tag{40}$$
$$= \sum_{r=0}^{\infty} C_{r+2}(r+1)(r+2)y^r \tag{41}$$

Plugging these into 34 we have (we'll drop the absolute value signs on $|m|$ to make the notation simpler; we can restore them at the end):

$$\sum_{r=0}^{\infty} C_{r+2}(r+1)(r+2)y^r + (2m+1)\sum_{r=0}^{\infty} C_r r y^{r-2} - \ldots \tag{42}$$

$$2\sum_{r=0}^{\infty} C_r r y^r + 2(\varepsilon - m - 1)\sum_{r=0}^{\infty} C_r y^r = 0 \tag{43}$$

The second sum in the first line is

$$\sum_{r=0}^{\infty} C_r r y^{r-2} = 0 + C_1 y^{-1} + 2C_2 + 3C_3 y + \ldots \tag{44}$$

$$= \sum_{r=-1}^{\infty} C_{r+2}(r+2)y^r \tag{45}$$

The sum thus becomes

$$(2m+1)C_1 y^{-1} + \sum_{r=0}^{\infty} y^r C_{r+2}(r+2)^2 + 2\sum_{r=0}^{\infty} y^r C_r[-r+\varepsilon - m - 1] = 0 \tag{46}$$

A basic theorem about power series is that if the sum of the series equals zero for all y, then the coefficient of each power must be zero. This shows that $C_1 = 0$ since otherwise the series would blow up as $y \to 0$. This results in a recursion relation for the C_r:

$$C_{r+2} = \frac{2(r+m+1-\varepsilon)}{(r+2)^2} C_r \tag{47}$$

Since $C_1 = 0$, all $C_r = 0$ for odd r. For large r we have

$$\frac{C_{r+2}}{C_r} \to \frac{2}{r} \tag{48}$$

If the series is allowed to be infinite, this leads to a divergent series as we can see from the following (based on Shankar's section 7.3). Suppose we look at $y^m e^{y^2}$, which clearly goes to infinity at large y (remember, m is positive). In series form this is

$$y^m e^{y^2} = \sum_{k=0}^{\infty} \frac{y^{2k+m}}{k!} \tag{49}$$

The coefficient C_n of y^n , with $n = 2k+m$ in this series is

$$C_n = \frac{1}{[(n-m)/2]!} \tag{50}$$

Similarly,

$$C_{n+2}=\frac{1}{\left[\left(n+2-m\right)/2\right]!} \tag{51}$$

The ratio is

$$\frac{C_{n+2}}{C_n}=\frac{\left[\left(n-m\right)/2\right]!}{\left[\left(n+2-m\right)/2\right]!} \tag{52}$$

$$=\frac{1}{\left(n-m\right)/2+1} \tag{53}$$

$$\to\frac{2}{n} \tag{54}$$

In other words, the coefficients of our series solution have the same behaviour 48 for large r as those in the series for $y^m e^{y^2}$. Referring back to 21, we see that this gives an overall behaviour for the radial function R of

$$R\to y^m e^{-y^2/2}y^m e^{y^2}=y^{2m}e^{y^2/2} \tag{55}$$

Thus if we allow the series for U to be infinite, the overall solution diverges, which is not acceptable. We therefore require that the series terminates at some finite value of r, and from 47 we see that this happens if

$$\varepsilon=r+m+1 \tag{56}$$

for some r. From the definition 8 this gives us the allowed values for the energy:

$$E=\hbar\omega\left(r+|m|+1\right) \tag{57}$$

$$=\hbar\omega\left(2k+|m|+1\right) \tag{58}$$

where the last line follows because r must be even. If

$$n\equiv 2k+|m| \tag{59}$$

then the allowed energies are

$$E=\hbar\omega\left(n+1\right) \tag{60}$$

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