UNCERTAINTY PRINCIPLE - CONDITION FOR MINIMUM UNCERTAINTY

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The generalized uncertainty principle relating two operators A and B is

$$\sigma_A^2\sigma_B^2 \ge \left(\frac{1}{2i}\left\langle [A,B]\right\rangle\right)^2 \tag{1}$$

where $[A,B]=AB-BA$ is the commutator of the operators.

This relation was derived using two inequalities. The first is the Schwarz inequality which, in bra-ket notation is

$$\left\langle f|f\right\rangle \left\langle g|g\right\rangle \ge \left|\left\langle f|g\right\rangle\right|^2 \tag{2}$$

The second inequality is a condition on a complex number $z=x+iy$:

$$|z|^2 \ge y^2 \tag{3}$$

It's interesting to see what happens if we require these two relations to be equalities rather than inequalities. This will give us a condition on the functions f and g that gives the minimum uncertainty between them.

In the case of the Schwarz inequality, we want

$$\left\langle f|f\right\rangle \left\langle g|g\right\rangle = \left|\left\langle f|g\right\rangle\right|^2 \tag{4}$$

To see what this implies, we can examine the proof of the Schwarz inequality. To this end, we'll introduce the function

$$|h\rangle \equiv |g\rangle - \frac{\langle f|g\rangle}{\langle f|f\rangle}|f\rangle \tag{5}$$

Since $\langle h|h\rangle \ge 0$

$$\langle h|h\rangle = \langle g|g\rangle - \frac{\langle f|g\rangle}{\langle f|f\rangle}\langle g|f\rangle - \frac{\langle g|f\rangle}{\langle f|f\rangle}\langle f|g\rangle + \left(\frac{|\langle f|g\rangle|}{\langle f|f\rangle}\right)^2\langle f|f\rangle \tag{6}$$

$$= \langle g|g\rangle - \frac{|\langle f|g\rangle|^2}{\langle f|f\rangle} \tag{7}$$

$$\geq 0 \tag{8}$$

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2 \tag{9}$$

In order for this inequality to be replaced with an equality, we would need

$$|h\rangle = 0 \tag{10}$$

$$|g\rangle = \frac{\langle f|g\rangle}{\langle f|f\rangle}|f\rangle \tag{11}$$

That is, the Schwarz inequality becomes an equality if one function is a scalar multiple of the other:

$$|g\rangle = c|f\rangle \tag{12}$$

where c is, in general, a complex scalar.

The second inequality (3 above) is an equality if $x = 0$ so that z is purely imaginary. In our original derivation of the uncertainty principle, we used $z = \langle f|g\rangle$, so if we're requiring equality, we get

$$z = \langle f|g\rangle \tag{13}$$

$$= c\langle f|f\rangle \tag{14}$$

and we require this to be purely imaginary. Since $\langle f|f\rangle$ is always real, this means that c must be imaginary, so we can write that the condition for equality is

$$|g\rangle = ia|f\rangle \tag{15}$$

with a a real scalar.

In terms of the operators A and B, referring back to our original derivation, this means that in order to get the minimum uncertainty, the wave function Ψ must satisfy

$$|(A - \langle A\rangle)\Psi\rangle = ia|(B - \langle B\rangle)\Psi\rangle \tag{16}$$

For example, if we consider the position and momentum operators, we get

$$\left(\frac{\hbar}{i}\frac{d}{dx}-\langle p\rangle\right)\mathbf{\Psi}=ia(x-\langle x\rangle)\mathbf{\Psi} \tag{17}$$

We can put the derivative on the LHS and everything else on the RHS (note that $\langle x\rangle$ and $\langle p\rangle$ are constants):

$$\frac{d\mathbf{\Psi}}{dx}=\frac{i}{\hbar}(ia(x-\langle x\rangle)+\langle p\rangle)\mathbf{\Psi} \tag{18}$$

$$\frac{d\mathbf{\Psi}}{\mathbf{\Psi}}=\left(-\frac{a}{\hbar}(x-\langle x\rangle)+\frac{i\langle p\rangle}{\hbar}\right)dx \tag{19}$$

$$\ln\Psi=-\frac{a}{2\hbar}(x-\langle x\rangle)^2+\frac{i\langle p\rangle x}{\hbar}+\ln A \tag{20}$$

$$\mathbf{\Psi}=Ae^{-a(x-\langle x\rangle)^2/2\hbar}e^{i\langle p\rangle x/\hbar} \tag{21}$$

In any stationary state $\langle p\rangle=0$, so any system in which there is a stationary state that has a gaussian wave function will have minimum position-momentum uncertainty. One case where this occurs is the ground state of the harmonic oscillator. (All higher states of the harmonic oscillator are gaussians multiplied by a higher-degree Hermite polynomial, so they aren't pure gaussians and will therefore have a higher uncertainty.)