

UNITARY MATRICES - SOME EXAMPLES

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Here are a few more results about unitary operators.

Shankar defines a unitary operator U as one where

$$UU^\dagger = I \quad (1)$$

From this we can derive the other condition by which they can be defined, namely that a unitary operator preserves the norm of a vector:

$$|Uv| = |v| \quad (2)$$

This follows, for if we define the effect of U by

$$|v'_1\rangle = U|v_1\rangle \quad (3)$$

then

$$\langle v'_1 | v'_1 \rangle = \langle Uv_1 | Uv_1 \rangle \quad (4)$$

$$= \langle v_1 | U^\dagger U v_1 \rangle \quad (5)$$

$$= \langle v_1 | v_1 \rangle \quad (6)$$

Thus $|v'_1|^2 = |v_1|^2$.

Theorem 1. *The product of two unitary operators U_1 and U_2 is unitary.*

Proof. Using Shankar's definition 1, we have

$$(U_1U_2)^\dagger U_1U_2 = U_2^\dagger U_1^\dagger U_1U_2 \quad (7)$$

$$= U_2^\dagger IU_2 \quad (8)$$

$$= U_2^\dagger U_2 \quad (9)$$

$$= I \quad (10)$$

□

Theorem 2. *The determinant of a unitary matrix U is a complex number with unit modulus.*

Proof. The determinant of a hermitian conjugate is the complex conjugate of the determinant of the original matrix, since $\det U = \det U^T$ (where the superscript T denotes the transpose) for any matrix, and the hermitian conjugate is the complex conjugate transpose. Therefore

$$\det(UU^\dagger) = [\det U][\det U]^* = \det I = 1 \quad (11)$$

Therefore $|\det U|^2 = 1$ as required. \square

Example 1. The rotation matrix $R\left(\frac{\pi}{2}\mathbf{i}\right)$ is unitary. We have

$$R\left(\frac{\pi}{2}\mathbf{i}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (12)$$

By direct calculation

$$RR^\dagger = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad (13)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad (14)$$

Example 2. Consider the matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad (15)$$

By calculating

$$UU^\dagger = \frac{1}{2} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \quad (16)$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I \quad (17)$$

Thus U is unitary, but because $U \neq U^\dagger$ it is not hermitian. Its determinant is

$$\det U = \left(\frac{1}{\sqrt{2}}\right)^2 (1 - i^2) = 1 \quad (18)$$

This is of the required form $e^{i\theta}$ with $\theta = 0$.

Example 3. Consider the matrix

$$U = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \quad (19)$$

$$UU^\dagger = \frac{1}{4} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} \quad (20)$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I \quad (21)$$

Thus U is unitary, but because $U \neq U^\dagger$ it is not hermitian. Its determinant is

$$\det U = \left(\frac{1}{2}\right)^2 [(1+i)^2 - (1-i)^2] \quad (22)$$

$$= i \quad (23)$$

This is of the required form $e^{i\theta}$ with $\theta = \frac{\pi}{2}$.