VECTOR OPERATORS - TRANSFORMATION UNDER ROTATION

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A vector operator $\mathbf{V}$ is defined as an operator whose components transform under rotation according to

$$U^{\dagger}[R]V_{i}U[R]=\sum_{j}R_{ij}V_{j} \tag{1}$$

where R is the rotation matrix in either 2 or 3 dimensions. We've seen that, for an infinitesimal rotation about an arbitrary axis $\delta\boldsymbol{\theta}$, a vector transforms like

$$\mathbf{V}\rightarrow\mathbf{V}+\delta\boldsymbol{\theta}\times\mathbf{V} \tag{2}$$

This can be written more compactly using the totally antisymmetric Levi-Civita tensor ε_{ijk}, where $\varepsilon_{123}=+1$ and the tensor is antisymmetric when any two indices are swapped. The component i of a cross product is

$$(\delta\boldsymbol{\theta}\times\mathbf{V})_{i}=\sum_{j,k}\varepsilon_{ijk}(\delta\theta)_{j}V_{k} \tag{3}$$

We get

$$\sum_{j}R_{ij}V_{j}=V_{i}+\sum_{j,k}\varepsilon_{ijk}(\delta\theta)_{j}V_{k} \tag{4}$$

The operator $U[R]$ is given by

$$U[R(\delta\boldsymbol{\theta})]=I-\frac{i}{\hbar}\delta\boldsymbol{\theta}\cdot\mathbf{L} \tag{5}$$

where $\mathbf{L}$ is the angular momentum. Plugging this into 1, we have, to first order in $\delta\boldsymbol{\theta}$ (remembering that the components of $\mathbf{L}$ do not commute with each other and, in general also do not commute with the components of $\mathbf{V}$):

$$\left(I+\frac{i}{\hbar}\delta\boldsymbol{\theta}\cdot\mathbf{L}\right)V_i\left(I-\frac{i}{\hbar}\delta\boldsymbol{\theta}\cdot\mathbf{L}\right)=V_i+\frac{i}{\hbar}\sum_j(\delta\theta_jL_j)V_i-\frac{i}{\hbar}V_i\sum_j(\delta\theta_jL_j) \tag{6}$$

$$=V_i+\frac{i}{\hbar}\sum_j\delta\theta_j[L_j,V_i] \tag{7}$$

Setting this equal to the RHS of 4 we have, equating coefficients of $\delta\theta_j$:

$$\frac{i}{\hbar}[L_j,V_i]=\sum_k\varepsilon_{ijk}V_k \tag{8}$$

$$[V_i,L_j]=i\hbar\sum_k\varepsilon_{ijk}V_k \tag{9}$$

With $\mathbf{V}=\mathbf{L}$, we regain the commutation relations for the components of angular momentum

$$[L_x,L_y]=i\hbar L_z \tag{10}$$

$$[L_y,L_z]=i\hbar L_x \tag{11}$$

$$[L_z,L_x]=i\hbar L_y \tag{12}$$

By the way, it is possible to write these commutation relations in the compact form

$$\mathbf{L}\times\mathbf{L}=i\hbar\mathbf{L} \tag{13}$$

This looks wrong if you're used to the standard definition of the cross product for vectors whose components are ordinary numbers, since for such a vector $\mathbf{a}$, we always have $\mathbf{a}\times\mathbf{a}=0$. However, if the components of the vector are *operators* that don't commute, then the result is not zero, as we can see:

$$(\mathbf{L}\times\mathbf{L})_i=\sum_{j,k}\varepsilon_{ijk}L_jL_k \tag{14}$$

If $i=x$, for example, then the sum on the RHS gives

$$(\mathbf{L}\times\mathbf{L})_x=\sum_{j,k}\varepsilon_{xjk}L_jL_k \tag{15}$$

$$=L_yL_z-L_zL_y \tag{16}$$

$$=[L_y,L_z] \tag{17}$$

From 13, this gives

$$[L_y, L_z] = i\hbar L_x \tag{18}$$