VECTOR SPACES AND LINEAR INDEPENDENCE - SOME EXAMPLES

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Given the axioms of a vector space, we can derive a few more properties. I'll use Shankar's notation for vectors, which is essentially Dirac's bra-ket notation.

Theorem 1. *The additive identity* 0 *is unique.*

Proof. Proof: (by contradiction). Suppose there are two distinct additive identities $|0\rangle$ and $|0'\rangle$. Then

$$\left|0'\right\rangle = \left|0'\right\rangle + |0\rangle \text{ (since } |0\rangle \text{is an additive identity)} \tag{1}$$

$$= |0\rangle + \left|0'\right\rangle \text{ (commutative addition)} \tag{2}$$

$$= |0\rangle \text{ (since } |0'\rangle \text{is an additive identity)} \tag{3}$$

□

Theorem 2. *Multiplication of any vector by the zero scalar gives the zero vector.*

Proof. We wish to show that $0|v\rangle = |0\rangle$ for all $v \in V$. We have

$$|0\rangle = (0+1)|v\rangle + |-v\rangle \tag{4}$$

$$= 0|v\rangle + |v\rangle + |-v\rangle \tag{5}$$

$$= 0|v\rangle + |0\rangle \tag{6}$$

$$= 0|v\rangle \tag{7}$$

where the third line follows because $|-v\rangle$ is the additive inverse of $|v\rangle$ and the last line follows because $|0\rangle$ is the additive identity vector. □

Theorem 3. $|-v\rangle = -|v\rangle$. *That is,* $-|v\rangle$ *is the additive inverse of* $|v\rangle$.

Proof. The negative of a vector v is multiplication of v by the scalar -1, so

$$|v\rangle+(-|v\rangle) = (1+(-1))|v\rangle \tag{8}$$
$$= 0|v\rangle \tag{9}$$
$$= |0\rangle \tag{10}$$

by theorem 2. Thus $-|v\rangle$ is an additive inverse of $|v\rangle$, so $-|v\rangle = |-v\rangle$. □

Theorem 4. *The additive inverse $|-v\rangle$ is unique.*

Proof. Suppose there is another vector $|w\rangle$ for which $|v\rangle+|w\rangle = |0\rangle$. By theorem 1, $|0\rangle$ is unique, so we must have $|v\rangle+|w\rangle = |v\rangle+|-v\rangle$. By theorem 3, this gives

$$|v\rangle-|v\rangle+|w\rangle = |-v\rangle \tag{11}$$
$$|0\rangle+|w\rangle = |-v\rangle \tag{12}$$
$$|w\rangle = |-v\rangle \tag{13}$$

where the third line follows because $|0\rangle$ is the additive identity. □

Example 1. Consider the set of all entities (a,b,c) where the entries are real numbers. Addition and scalar multiplication are defined as

$$(a,b,c)+(d,e,f) \equiv (a+d,b+e,c+f) \tag{14}$$
$$\alpha(a,b,c) \equiv (\alpha a,\alpha b,\alpha c) \tag{15}$$

The null vector is

$$|0\rangle = (0,0,0) \tag{16}$$

The inverse of (a,b,c) is $(-a,-b,-c)$. As the set is closed under addition and scalar multiplication it is a vector space. However, a subset such as $(a,b,1)$ is *not* a vector space since it is not closed under addition or scalar multiplication:

$$(a,b,1)+(d,e,1) = (a+d,b+e,2) \tag{17}$$
$$2(a,b,1) = (2a,2b,2) \tag{18}$$

Neither of the vectors on the RHS are of the form $(a,b,1)$ so they don't lie in the set.

Example 2. The set of all functions $f(x)$ defined on an interval $0 \le x \le L$ form a vector space if we define addition as pointwise addition $f+g = f(x)+g(x)$ for all x, and scalar multiplication by a as $af(x)$.

Some subsets of this vector space are also vector spaces. For example the set of all functions that satsify $f(0)=f(L)=0$ is a vector space, because the sum of any two such functions also satisfies $(f+g)(0)=(f+g)(L)=0$, and scalar multiplication leaves the endpoints at 0 as well.

The subset of periodic functions $f(0)=f(L)$ (not necessarily equal to 0) is also a vector space. Adding any two functions from this subset gives a sum such that

$$f(0)+g(0)=f(L)+g(L) \tag{19}$$
$$(f+g)(0)=(f+g)(L) \tag{20}$$

Multiplying by a scalar gives

$$a(f(0)+g(0))=a(f(L)+g(L)) \tag{21}$$
$$a(f+g)(0)=a(f+g)(L) \tag{22}$$

However, a subset such as all functions with $f(0)=4$ is not a vector space, since adding two such functions gives a sum with $(f+g)(0)=8$, and multiplying by a scalar gives a function with $af(0)=4a$, neither of which is in the subset.

Now a couple of examples of linear independence.

Example 3. We have three vectors from the vector space of real 2×2 matrices:

$$|1\rangle=\left[\begin{array}{c}01\\00\end{array}\right] \tag{23}$$
$$|2\rangle=\left[\begin{array}{c}11\\01\end{array}\right] \tag{24}$$
$$|3\rangle=\left[\begin{array}{c}-2-1\\0-2\end{array}\right] \tag{25}$$

These are not linearly independent, because $|3\rangle=|1\rangle-2|2\rangle$.

Example 4. We have 3 row vectors

$$|1\rangle=\left[\begin{array}{cc}1 & 10\end{array}\right] \tag{26}$$
$$|2\rangle=\left[\begin{array}{cc}1 & 01\end{array}\right] \tag{27}$$
$$|3\rangle=\left[\begin{array}{cc}3 & 21\end{array}\right] \tag{28}$$

These are linearly dependent, since $|3\rangle=2|1\rangle+|2\rangle$.

Now we look at the 3 vectors

$$|1\rangle = \begin{bmatrix} 1 & 10 \end{bmatrix} \tag{29}$$
$$|2\rangle = \begin{bmatrix} 1 & 01 \end{bmatrix} \tag{30}$$
$$|3\rangle = \begin{bmatrix} 0 & 11 \end{bmatrix} \tag{31}$$

We can show that these are linearly independent by attempting to solve the equation

$$0 = a|1\rangle + b|2\rangle + c|3\rangle \tag{32}$$

Looking at each component, we have

$$a + b = 0 \tag{33}$$
$$a + c = 0 \tag{34}$$
$$b + c = 0 \tag{35}$$

Solving the last two equations for a and b in terms of c and substituting into the first equation, we get

$$-2c = 0 \tag{36}$$
$$c = 0 \tag{37}$$

Thus we find that the only solution is $a = b = c = 0$, which proves linear independence.