

WKB APPROXIMATION AND THE HYDROGEN ATOM

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Post date: 17 October 2021.

As another example of applying the WKB approximation to a 3-d problem with a spherically symmetric potential, we'll look at the radial equation for the hydrogen atom.

The general wave function can be written as the product of a radial function $R(r)$ and a spherical harmonic $Y(\theta, \phi)$.

With the substitution $u(r) \equiv rR(r)$ the radial equation for hydrogen can be written

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \left(-\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right) u = Eu \quad (1)$$

The effective potential is the term in parentheses:

$$V(r) = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \quad (2)$$

and has the form of a well with sloping sides on both sides, at points r_1 and r_2 defined by the roots of

$$E = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r} + \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \quad (3)$$

In this case, the WKB equations satisfy the condition

$$\int_{r_1}^{r_2} p(r) dr = \left(n - \frac{1}{2} \right) \pi \hbar \quad (4)$$

Plugging in the formulas for the hydrogen atom we get

$$\int_{r_1}^{r_2} p(r) dr = \sqrt{2m} \int_{r_1}^{r_2} \sqrt{E + \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}} dr \quad (5)$$

$$= \sqrt{-2mE} \int_{r_1}^{r_2} \frac{1}{r} \sqrt{-r^2 - \frac{e^2}{4\pi\epsilon_0 E} r + \frac{\hbar^2}{2mE} l(l+1)} dr \quad (6)$$

where we've factored out $-E$ (since for a bound state, $E < 0$). We can simplify the notation by defining the positive quantities

$$B \equiv -\frac{e^2}{4\pi\epsilon_0 E} \quad (7)$$

$$C \equiv -\frac{\hbar^2}{2mE} l(l+1) \quad (8)$$

The turning points r_1 and r_2 can be found from the roots of the term inside the square root in 6:

$$r_1 = \frac{B - \sqrt{B^2 - 4C}}{2} \quad (9)$$

$$r_2 = \frac{B + \sqrt{B^2 - 4C}}{2} \quad (10)$$

so we can factor the quadratic to get

$$\int_{r_1}^{r_2} p(r) dr = \sqrt{-2mE} \int_{r_1}^{r_2} \frac{1}{r} \sqrt{(r-r_1)(r_2-r)} dr \quad (11)$$

$$= \sqrt{-2mE} \frac{\pi}{2} (\sqrt{r_2} - \sqrt{r_1})^2 \quad (12)$$

$$= \sqrt{-2mE} \frac{\pi}{2} (r_1 + r_2 - 2\sqrt{r_1 r_2}) \quad (13)$$

$$= \sqrt{-2mE} \frac{\pi}{2} (B - 2\sqrt{C}) \quad (14)$$

$$= \sqrt{-2mE} \frac{\pi}{2} \left(-\frac{e^2}{4\pi\epsilon_0 E} - 2\sqrt{-\frac{\hbar^2}{2mE} l(l+1)} \right) \quad (15)$$

$$= \frac{\pi}{2} \left(-\frac{e^2 \sqrt{2m}}{4\pi\epsilon_0 \sqrt{-E}} - 2\hbar \sqrt{l(l+1)} \right) \quad (16)$$

$$= \left(n - \frac{1}{2} \right) \pi \hbar \quad (17)$$

where we've used the integral given in Griffiths's question to get the second line, and the last line uses 4. Solving 16 for E gives the WKB energy levels:

$$E = - \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{m}{2\hbar^2 \left(n - \frac{1}{2} + \sqrt{l(l+1)} \right)^2} \quad (18)$$

$$= \frac{-13.6 \text{ eV}}{\left(n - \frac{1}{2} + \sqrt{l(l+1)} \right)^2} \quad (19)$$

since the ground state of hydrogen is given by the Bohr formula

$$E_0 = - \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 \frac{m}{2\hbar^2} = -13.6 \text{ eV} \quad (20)$$

COMMENTS

From Steve Goldstein, 28 Feb 2022, 16:29:

Does equation 2 describe the effective potential energy, not the effective potential?

Reply from growscience, 1 Mar 2022, 10:14:

It's actually potential energy, but most QM books refer to V as just 'potential'. This is explained in the footnote in Griffiths's QM book at the beginning of Chapter 2:

"It is tiresome to keep saying 'potential energy function' so most people just call V the 'potential', even though this invites occasional confusion with *electric potential*, which is actually potential energy per unit charge."