

## WKB APPROXIMATION AT A TURNING POINT WITH DECREASING POTENTIAL

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In the last post, we worked out the WKB approximation on either side of a turning point where the potential is increasing. We can do the same analysis for a turning point where the potential is decreasing. The calculations are almost exactly the same as in the first case.

The wave function has the form

$$\psi(x) \approx \frac{1}{\sqrt{p(x)}} \left( B e^{i \int p \, dx / \hbar} + C e^{-i \int p \, dx / \hbar} \right) \quad (1)$$

if the particle's energy  $E > V(x)$ , where

$$p(x) = \sqrt{2m(E - V(x))} \quad (2)$$

is the momentum of the particle. In the tunneling case, where  $E < V(x)$ , the WKB approximation gives

$$\psi(x) \approx \frac{1}{\sqrt{|p|}} \left( D e^{-\frac{1}{\hbar} \int |p(x)| \, dx} + F e^{\frac{1}{\hbar} \int |p(x)| \, dx} \right) \quad (3)$$

In the decreasing potential case, we have  $E < V(x)$  for  $x < x_1$  and  $E > V(x)$  for  $x > x_1$ . As before, we'll move the turning point so that  $x_1 = 0$  and shift it back after the analysis. In this case, the limits on the integrals above give the results

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{|p|}} \left( D e^{-\frac{1}{\hbar} \int_x^0 |p(x')| \, dx'} + F e^{\frac{1}{\hbar} \int_x^0 |p(x')| \, dx'} \right) & x < 0 \\ \frac{1}{\sqrt{p(x)}} \left( B e^{i \int_0^x p(x') \, dx' / \hbar} + C e^{-i \int_0^x p(x') \, dx' / \hbar} \right) & x > 0 \end{cases} \quad (4)$$

As the integrand for  $x < 0$  is non-negative, we must have  $F = 0$  to keep  $\psi$  finite as  $x \rightarrow -\infty$ , so we have

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{|p|}} \left( D e^{-\frac{1}{\hbar} \int_x^0 |p(x')| \, dx'} \right) & x < 0 \\ \frac{1}{\sqrt{p(x)}} \left( B e^{i \int_0^x p(x') \, dx' / \hbar} + C e^{-i \int_0^x p(x') \, dx' / \hbar} \right) & x > 0 \end{cases} \quad (5)$$

As before, we assume that around  $x = 0$

$$V(x) \approx E + V'(0)x \quad (6)$$

where now  $V'(0) < 0$  since the potential is decreasing. We now define the patching function  $\psi_p$  that satisfies the Schrödinger equation with this approximate potential near  $x = 0$ .

$$\frac{d^2\psi_p}{dx^2} = \frac{2mV'(0)}{\hbar^2}x\psi_p \quad (7)$$

$$= -\frac{2m|V'(0)|}{\hbar^2}x\psi_p \quad (8)$$

By defining

$$\alpha \equiv \left( \frac{2m|V'(0)|}{\hbar^2} \right)^{1/3} \quad (9)$$

$$z \equiv -\alpha x \quad (10)$$

we end up with Airy's equation

$$\frac{d^2\psi_p}{dz^2} = z\psi_p \quad (11)$$

As we saw earlier, this has the general solution

$$\psi_p(z) = aAi(z) + bBi(z) \quad (12)$$

$$\psi_p(x) = aAi(-\alpha x) + bBi(-\alpha x) \quad (13)$$

It is this function that we need to match up to 5.

To do this, we make a further assumption that the overlap regions (where we're trying to match the WKB functions with  $\psi_p$ ) are far enough from  $x = 0$  that we can use the asymptotic forms of the Airy functions. These forms are

$$Ai(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin \left[ \frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-2z^{3/2}/3} & z \gg 0 \end{cases} \quad (14)$$

$$Bi(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos \left[ \frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{\sqrt{\pi}z^{1/4}} e^{2z^{3/2}/3} & z \gg 0 \end{cases} \quad (15)$$

The forms can't be used if we get too close to  $x = 0$ , but the hope is that there is still some range of  $x$  where both the WKB wave function and

$\psi_p$  are reasonable approximations to the true wave function. Under this assumption, we have from 13 for  $z = -\alpha x \gg 0$ , or  $x \ll 0$ :

$$\psi_p(x) = \frac{a}{2\sqrt{\pi}z^{1/4}}e^{-2z^{3/2}/3} + \frac{b}{\sqrt{\pi}z^{1/4}}e^{2z^{3/2}/3} \quad (16)$$

$$= \frac{a}{2\sqrt{\pi}(-\alpha x)^{1/4}}e^{-2(-\alpha x)^{3/2}/3} + \frac{b}{\sqrt{\pi}(\alpha x)^{1/4}}e^{2(-\alpha x)^{3/2}/3} \quad (17)$$

With the potential 6, the WKB function 5 for  $x < 0$  is

$$p(x) = \sqrt{2m(E - V(x))} \quad (18)$$

$$= \sqrt{-2mV'(0)x} \quad (19)$$

$$= \hbar\sqrt{x}\alpha^{3/2} \quad (20)$$

$$\psi(x) = \frac{1}{\sqrt{|p|}}De^{-\frac{1}{\hbar}\int_x^0 |p(x')|dx'} \quad (21)$$

$$\int_0^x |p(x')|dx' = \sqrt{-2mV'(0)} \int_x^0 \sqrt{-x'}dx' \quad (22)$$

$$= \sqrt{2m|V'(0)|} \frac{2}{3} (-x)^{3/2} \quad (23)$$

$$= \frac{2}{3}\hbar(-\alpha x)^{3/2} \quad (24)$$

$$\psi(x) = \frac{D}{\sqrt{\hbar}\alpha^{3/4}(-x)^{1/4}}e^{-2(-\alpha x)^{3/2}/3} \quad (25)$$

Matching this up with 17, we see that  $b = 0$  and

$$\frac{a}{2\sqrt{\pi}\alpha^{1/4}} = \frac{D}{\sqrt{\hbar}\alpha^{3/4}} \quad (26)$$

$$a = 2\sqrt{\frac{\pi}{\hbar}\alpha}D \quad (27)$$

The patching function must therefore have the form

$$\psi_p(x) = 2\sqrt{\frac{\pi}{\hbar}\alpha}De^{-2(-\alpha x)^{3/2}/3} \quad (28)$$

for  $\alpha x \ll 0$ .

We can do the same calculation for the overlap region for  $x > 0$ , where  $E > V(x)$ . Since  $x > 0$ ,  $z = -\alpha x < 0$  and we can use the asymptotic form of  $Ai(x)$  for large negative  $x$  and we get

$$\psi_p(x) = \frac{a}{\sqrt{\pi}(\alpha x)^{1/4}} \sin \left[ \frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4} \right] \quad (29)$$

For the WKB function, we start with 5

$$p(x) = \sqrt{-2mV'(0)x} \quad (30)$$

$$= \hbar\alpha^{3/2}\sqrt{x} \quad (31)$$

$$\int_0^x p(x') dx' = \frac{2}{3}\hbar(\alpha x)^{3/2} \quad (32)$$

$$\psi(x) = \frac{1}{\sqrt{\hbar}\alpha^{3/4}(x)^{1/4}} \left( B e^{2i(\alpha x)^{3/2}/3} + C e^{-2i(\alpha x)^{3/2}/3} \right) \quad (33)$$

We can now compare this with 29 by writing the sine in terms of exponentials

$$\psi_p(x) = \frac{a}{2i\sqrt{\pi}(\alpha x)^{1/4}} \left[ e^{i\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right]} - e^{-i\left[\frac{2}{3}(\alpha x)^{3/2} + \frac{\pi}{4}\right]} \right] \quad (34)$$

Comparing terms, we have

$$\frac{B}{\sqrt{\hbar}\alpha^{3/4}} = \frac{ae^{i\pi/4}}{2i\sqrt{\pi}\alpha^{1/4}} \quad (35)$$

$$B = \frac{ae^{i\pi/4}\sqrt{\hbar}\alpha}{2i\sqrt{\pi}} \quad (36)$$

$$= -ie^{i\pi/4}D \quad (37)$$

$$\frac{C}{\sqrt{\hbar}\alpha^{3/4}} = -\frac{ae^{-i\pi/4}}{2i\sqrt{\pi}\alpha^{1/4}} \quad (38)$$

$$C = -\frac{ae^{-i\pi/4}\sqrt{\hbar}\alpha}{2i\sqrt{\pi}} \quad (39)$$

$$= ie^{-i\pi/4}D \quad (40)$$

We now have the constants  $B$  and  $C$ , belonging to the WKB wave function for  $x > 0$ , in terms of the constant  $D$  belonging to the WKB function for  $x < 0$ , which is what we were after. Note that the WKB functions are still valid only for values of  $x$  that maintain a respectable distance from the turning point; all we have done is connected the approximations on either side of the turning point. The final form of the WKB function is, for  $x > 0$ :

$$\psi(x) \cong \frac{-iD}{\sqrt{p(x)}} \left( e^{i(\int p dx/\hbar + \pi/4)} - e^{-i(\int p dx/\hbar + \pi/4)} \right) \quad (41)$$

$$= \frac{2D}{\sqrt{p(x)}} \frac{1}{2i} \left( e^{i(\int p dx/\hbar + \pi/4)} - e^{-i(\int p dx/\hbar + \pi/4)} \right) \quad (42)$$

$$= \frac{2D}{\sqrt{p(x)}} \sin \left[ \int_0^x p dx'/\hbar + \pi/4 \right] \quad (43)$$

And for  $x < 0$

$$\psi(x) \cong \frac{D}{\sqrt{|p(x)|}} \exp \left[ - \int_x^0 |p(x')| dx'/\hbar \right] \quad (44)$$

If the turning point is at the general location of  $x_1$ , we can just replace the 0 in the limits of the integrals by  $x_1$ . The only difference between the cases of increasing and decreasing potentials is that the limits of integration are reversed.

#### PINGBACKS

Pingback: WKB approximation of the harmonic oscillator

Pingback: WKB approximation for a barrier with sloping sides

Pingback: WKB approximation of a double potential well - turning points