

WKB APPROXIMATION OF THE HARMONIC OSCILLATOR

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We've seen that if we apply the WKB approximation to a region where the potential is increasing we get

$$\psi(x) \approx \begin{cases} \frac{2D}{\sqrt{p(x)}} \sin \left[\int_x^{x_2} p dx' / \hbar + \pi/4 \right] & x < x_2 \\ \frac{D}{\sqrt{|p(x)|}} \exp \left[- \int_{x_2}^x |p(x')| dx' / \hbar \right] & x > x_2 \end{cases} \quad (1)$$

where x_2 is the turning point, where the particle's energy $E = V(x_2)$ with $V'(x_2) > 0$ and D is a normalization constant.

At a point x_1 where the potential is decreasing and $E = V(x_1)$ with $V'(x_1) < 0$ we get

$$\psi(x) \approx \begin{cases} \frac{2D'}{\sqrt{p(x)}} \sin \left[\int_{x_1}^x p dx' / \hbar + \pi/4 \right] & x > x_1 \\ \frac{D'}{\sqrt{|p(x)|}} \exp \left[- \int_x^{x_1} |p(x')| dx' / \hbar \right] & x < x_1 \end{cases} \quad (2)$$

where D' is another normalization constant.

If we're applying the WKB approximation to a potential well where the energy E is greater than $V(x)$ only in the region $x_1 < x < x_2$ then the wave function in this region can be written as the sine function from either of these two forms, that is

$$\psi(x) \approx \frac{2D}{\sqrt{p(x)}} \sin \left[\int_x^{x_2} p dx' / \hbar + \pi/4 \right] \quad (3)$$

$$\approx \frac{2D'}{\sqrt{p(x)}} \sin \left[\int_{x_1}^x p dx' / \hbar + \pi/4 \right] \quad (4)$$

Because the sine is an odd function, we can write the second form as

$$\psi(x) \approx - \frac{2D'}{\sqrt{p(x)}} \sin \left[- \int_{x_1}^x p dx' / \hbar - \pi/4 \right] \quad (5)$$

The zeroes of the sines must match up between these two forms which means the arguments of these two sines must be equal up to a multiple of π :

$$\frac{1}{\hbar} \int_x^{x_2} p dx' + \frac{\pi}{4} = -\frac{1}{\hbar} \int_{x_1}^x p dx' - \frac{\pi}{4} + n\pi \quad (6)$$

$$\left(\int_{x_1}^x + \int_x^{x_2} \right) p dx' = \left(n - \frac{1}{2} \right) \pi \hbar \quad (7)$$

$$\int_{x_1}^{x_2} p dx = \left(n - \frac{1}{2} \right) \pi \hbar \quad (8)$$

where $n = 1, 2, 3, \dots$ (we have to start at $n = 1$ rather than $n = 0$ to keep the integral positive).

We can make the two approximate forms of ψ equal if we also set $D' = -D$.

Example 1. As a simple illustration of this, we consider the harmonic oscillator, with a potential

$$V(x) = \frac{1}{2} kx^2 \quad (9)$$

$$p(x) = \sqrt{2m \left(E - \frac{1}{2} kx^2 \right)} \quad (10)$$

In this case, the turning points are

$$x_1 = -\sqrt{\frac{2E}{k}} \quad (11)$$

$$x_2 = \sqrt{\frac{2E}{k}} \quad (12)$$

and

$$\int_{x_1}^{x_2} p dx = \sqrt{2m} \int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \sqrt{E - \frac{kx^2}{2}} dx \quad (13)$$

$$= \pi E \sqrt{\frac{m}{k}} \quad (14)$$

$$= \left(n - \frac{1}{2} \right) \pi \hbar \quad (15)$$

$$E = \left(n - \frac{1}{2} \right) \sqrt{\frac{k}{m}} \hbar \quad (16)$$

$$= \left(n - \frac{1}{2} \right) \hbar \omega \quad (17)$$

Since n starts at 1, this gives the same sequence of energies as the exact analysis.

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