A FLAWED THEORY OF GRAVITY

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In Newtonian gravitational theory, the potential Φ satisfies Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho \tag{1}$$

where G is the gravitational constant and ρ is the mass density. This is the gravitational analogue of Poisson's equation in electrostatics. Continuing the analogy with electrostatics, we can define a gravitational field **G** as the force per unit mass and, since gravity is an inverse square force, we can write $\mathbf{G} = -\nabla\Phi$. The force per unit mass is the acceleration so

$$\frac{d\overrightarrow{v}}{dt} = -\nabla\Phi \tag{2}$$

where $\overrightarrow{v}$ is the 3-d velocity vector.

In the case of flat space, we've seen that the derivative of a four-vector is also a four-vector, so in that limit (which isn't quite accurate, since the presence of mass invalidates special relativity, but if the mass density is small, we should be OK) we can try to generalize the equations above to four-vector equations:

$$\partial^i\partial_i\Phi = 4\pi G\rho \tag{3}$$

$$\frac{du_i}{d\tau} = -\partial_i\Phi \tag{4}$$

These reduce to the above equations if Φ is independent of time (since the ∂_t term is the derivative with respect to time in the implied sums).

However, since the square of the four-velocity is an invariant: $u^iu_i = -1$, we must have

$$\frac{d}{d\tau}\left(u^iu_i\right) = 2u^i\frac{du_i}{d\tau} = 0 \tag{5}$$

so if we multiply the second equation above by u^i and sum, we get

$$u^i \frac{du_i}{d\tau} = -u^i \partial_i \Phi = 0 \tag{6}$$

Since the four-velocity can be anything, this implies that $\partial_i \Phi = 0$, which means that $\partial^i \partial_i \Phi = 0$, contradicting the first of the equations above (unless $\rho = 0$ which isn't a very interesting case).

This problem can be fixed by adding a term to the derivative of u^i:

$$\frac{du_i}{d\tau} = -\partial_i \Phi - u_i u^a \partial_a \Phi \tag{7}$$

This works because if we now multiply both sides by u^i and sum, we get

$$\begin{aligned} u^i \frac{du_i}{d\tau} &= -u^i \partial_i \Phi - u^i u_i u^a \partial_a \Phi & (8) \\ &= -u^i \partial_i \Phi - (-1)\, u^a \partial_a \Phi & (9) \\ &= 0 & (10) \end{aligned}$$

using $u^i u_i = -1$. In the limit of small four-velocity, the $-u_i u^a \partial_a \Phi$ term can be neglected because it is second order in u^i, so it does preserve the Newtonian limit.

If we consider the equation $\partial^i \partial_i \Phi = 4\pi G \rho$ in empty space (where $\rho = 0$) and write it out in cartesian coordinates, we get

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 0 \tag{11}$$

This is the homogeneous wave equation, with a propagation speed of $c = 1$. This could be interpreted as a prediction of gravitational waves that travel at the speed of light.

If space contains mass, this equation becomes

$$\frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = 4\pi G \rho \tag{12}$$

which is the inhomogeneous wave equation, with $4\pi G \rho$ being the source term.

If the equations involving four-vectors above really are tensor equations, they should be invariant under a transformation of coordinates. In particular, if we transform the scalar ρ, it should remain the same in the new frame. The problem is that, although *mass* is invariant, mass *density* isn't. This is for much the same reason as was responsible for charge density changing with reference frame. If an object has a mass density of ρ at rest, then as seen from a moving frame, its length contracts due to Lorentz contraction, so its density increases by a factor of $\gamma = 1/\sqrt{1 - v^2}$. Thus these equations are not true tensor equations, as they apply only in one reference frame. We'll still need to learn general relativity properly after all.