AREA AND VOLUME IN SPACETIME

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Given a metric, we can calculate distances, areas and volumes in spacetime. For this example, we'll use a diagonal metric, which means that the basis vectors in these coordinates are orthogonal.

We're given a metric in the form

$$ds^2 = -\left(1 - Ar^2\right)^2 dt^2 + \left(1 - Ar^2\right)^2 dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{1}$$

where A is a constant, and (r, θ, ϕ) are the usual spherical coordinates in space.

We first find the distance at constant t along a radial line from $r = 0$ to $r = R$ for some constant R. Along a radial line at constant t, we have $dt = d\theta = d\phi = 0$ and the infinitesimal distance is

$$ds = \left(1 - Ar^2\right) dr \tag{2}$$

There appears to be a bit of a catch here, however. We need the infinitesimal length to be positive, so this distance is valid only if $r < \frac{1}{\sqrt{A}}$. For larger values of r, we would need to use the absolute value $\left|1 - Ar^2\right| = Ar^2 - 1$.

The radial distance is, for $R < \frac{1}{\sqrt{A}}$

$$D_1 = \int_0^R \left(1 - Ar^2\right) dr = R\left(1 - \frac{1}{3}AR^2\right) \tag{3}$$

For $R > \frac{1}{\sqrt{A}}$, we have

$$D_1 = \int_0^{1/\sqrt{A}} \left(1 - Ar^2\right) dr + \int_{1/\sqrt{A}}^R \left(Ar^2 - 1\right) dr \tag{4}$$

$$= \frac{2}{3\sqrt{A}} + \frac{1}{3A}\left(A^2R^3 - 3AR + 2\sqrt{A}\right) \tag{5}$$

We can't take the limit of this expression as $A \to 0$, since it is valid only for $R > \frac{1}{\sqrt{A}}$, and this would require $R \to \infty$.

The area of a surface defined by $r = R =$ constant is found by setting $dt = dr = 0$ and integrating over the angular variables. The area element is given by

$$dA = \sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta\, d\phi = \sqrt{R^2 \times R^2 \sin^2\theta} = R^2 \sin\theta \tag{6}$$

Note that we set $r = R$ here since we're considering a constant value of r. The area is then

$$D_2 = \int_0^\pi d\theta \int_0^{2\pi} d\phi\, R^2 \sin\theta = 4\pi R^2 \tag{7}$$

Thus a sphere in this geometry has the same area as an ordinary sphere in 3-space.

The volume of such a sphere is different, however, because we now must integrate over r as well. The volume element is

$$\begin{aligned} d\mathcal{V} &= \sqrt{g_{rr}g_{\theta\theta}g_{\phi\phi}}dr\, d\theta\, d\phi && (8)\\ &= \sqrt{\left(1 - Ar^2\right)^2 r^2 r^2 \sin^2\theta}dr\, d\theta\, d\phi && (9)\\ &= \left(1 - Ar^2\right) r^2 \sin\theta\, dr\, d\theta\, d\phi && (10) \end{aligned}$$

The 3-volume of the sphere is then

$$\begin{aligned} D_3 &= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \left(1 - Ar^2\right) r^2 \sin\theta && (11)\\ &= \frac{4}{3}\pi R^3 - \frac{4}{5}\pi A R^5 && (12)\\ &= \frac{4\pi R^3}{3}\left(1 - \frac{3AR^2}{5}\right) && (13) \end{aligned}$$

If $A = 0$, we get the usual formula for the volume of a sphere in flat space. In this geometry, however, the sphere has a smaller volume than in flat space. As with the linear distance, this is valid only for $R < \frac{1}{\sqrt{A}}$. For $R > \frac{1}{\sqrt{A}}$, we have

$$D_3 = \int_0^{1/\sqrt{A}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \left(1 - Ar^2\right) r^2 \sin\theta + \int_{1/\sqrt{A}}^{R} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \left(1 - Ar^2\right) r^2 \sin\theta \tag{14}$$

$$= \frac{4\pi\left(3A^{\frac{5}{2}}R^5 - 5A^{\frac{3}{2}}R^3 + 4\right)}{15A^{\frac{3}{2}}} \tag{15}$$

where I used Maple to do the integrals.

To find the four-dimensional volume enclosed by a tube bounded by a sphere of radius R and times $t = 0$ and $t = T$, we integrate the 4-dim volume element

$$d\mathcal{V}_4 = \sqrt{-g_{tt}g_{rr}g_{\theta\theta}g_{\phi\phi}}\,dt\;dr\;d\theta\;d\phi \tag{16}$$

$$= \left(1 - Ar^2\right)^2 r^2 \sin\theta\;dt\;dr\;d\theta\;d\phi \tag{17}$$

where the minus sign is introduced because g_{tt} is negative in 1. This time we have

$$D_4 = \int_0^T dt \int_0^R dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \left(1 - Ar^2\right)^2 r^2 \sin\theta \tag{18}$$

$$= \frac{4}{7}A^2\pi R^7 T - \frac{8}{5}A\pi R^5 T + \frac{4}{3}\pi R^3 T \tag{19}$$

This time we don't have the problem of a negative infinitesimal interval, since the factor $\left(1 - Ar^2\right)^2$ is always positive.

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