

AREA OF UNIT SPHERES IN HIGHER DIMENSIONS

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We now consider a general formula for calculating the area of a unit sphere S^d of d dimensions embedded in Euclidean space E^{d+1} of one higher dimension than the sphere. Familiar examples are S^1 (a circle) embedded in E^2 (a plane), and S^2 (the usual sphere) embedded in 3-d Euclidean space E^3 .

The general formula is an integral over the d angle variables that define a point on the surface of S^d . If we call the generalized area A^d , then

$$A^d = \int_0^{2\pi} d\theta_d \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{d-1} \sqrt{g} \quad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$. Note that the last angle θ_d runs from 0 to 2π while all the other angles run from 0 to π . For example, with $d = 1$, the circle has only one angle variable which runs from 0 to 2π . With an ordinary sphere S^2 , the polar angle (usually written as $\theta_1 = \theta$) runs from 0 to π while the azimuthal angle $\theta_2 = \phi$ runs from 0 to 2π .

We can get the metric tensor from the general formula for ds_d^2 .

$$ds_d^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \dots + \sin^2 \theta_1 \dots \sin^2 \theta_{d-1} d\theta_d^2 \quad (2)$$

From this we see that $g_{\mu\nu}$ is diagonal, with diagonal entry given by $g_{11} = 0$ and, for $i > 1$

$$g_{ii} = \prod_{j=1}^{i-1} \sin^2 \theta_j \quad (3)$$

Zee asks us to check the answer for $d = 2$ and $d = 3$ but I think he means $d = 1$ and $d = 2$ to get the circle and ordinary sphere, as described above. For $d = 1$ we have $g_{11} = 1$ so the metric matrix is just a 1×1 matrix with determinant $g = 1$, and

$$A^1 = \int_0^{2\pi} d\theta_1 = 2\pi \quad (4)$$

which is the circumference of a unit circle.

For $d = 2$, we have

$$A^2 = \int_0^{2\pi} d\theta_2 \int_0^\pi d\theta_1 \sqrt{g} \quad (5)$$

In this case,

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 \theta_1 \end{bmatrix} \quad (6)$$

so $g = \sin^2 \theta_1$ and we have

$$A^2 = \int_0^{2\pi} d\theta_2 \int_0^\pi d\theta_1 \sin \theta_1 = 4\pi \quad (7)$$

which is the area of a unit sphere embedded in 3-d Euclidean space.

For $d = 3$, we have, using 2

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \theta_1 & 0 \\ 0 & 0 & \sin^2 \theta_1 \sin^2 \theta_2 \end{bmatrix} \quad (8)$$

$$\sqrt{g} = \sin^2 \theta_1 \sin \theta_2 \quad (9)$$

The area is then

$$A^3 = \int_0^{2\pi} d\theta_3 \int_0^\pi d\theta_2 \sin \theta_2 \int_0^\pi d\theta_1 \sin^2 \theta_1 \quad (10)$$

$$= 2\pi^2 \quad (11)$$

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